Extending Wilf-Equivalence Results Among Partial Shuffles

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Joint work with Michael Albert and Dominic Searles.

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Permutation Patterns July 2025

- Hamaker, Pawlowski, and Sagan (2019) investigated a certain *quasisymmetric function* indexed by descent sets of permutations in avoidance classes.
- They determined for which subsets of $\mathfrak{S}_3$ this function is *symmetric* and *Schur-positive*, and provided several conjectures.
- Bloom and Sagan (2020) introduced *partial shuffles* to prove one of these conjectures.

Partial shuffle definition

For $\ell \in \mathbb{N}$, let $[\ell] = \{1, 2, \ldots, \ell\}$. Pick $a \in [\ell]$ and b with $a + b = \ell$.

Definition

The partial shuffle $\Pi(a, b)$ is the set of permutations of $[\ell]$ where every element except a is increasing, *excluding* the strictly increasing permutation $12\ldots\ell$.

Example

Let $\ell = 5$ and $a = 3$. Then

$$\begin{aligned} \Pi(3, 2) &= \{1245 \sqcup 3\} - 12345 \\ &= \{12453, 12435, 13245, 31245\} \end{aligned}$$

Examples of partial shuffles

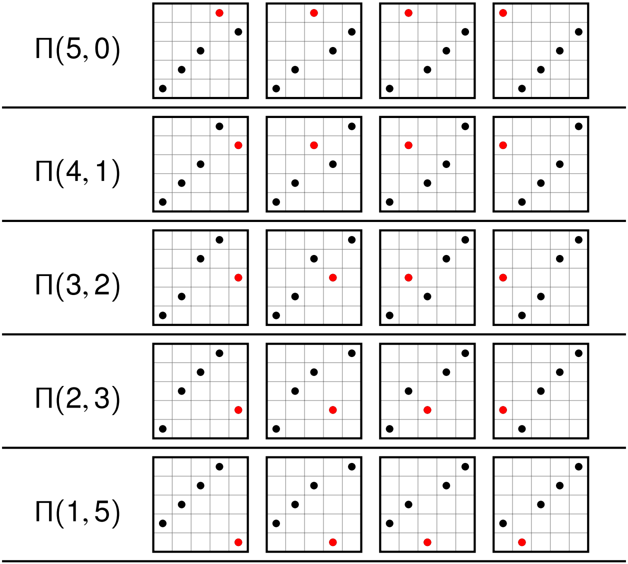


Figure: The five partial shuffles of length 5

A family of Wilf-equivalences

Theorem (Bloom & Sagan 2019, Albert & Searles & SH 2025+)

If $a + b = c + d$, then

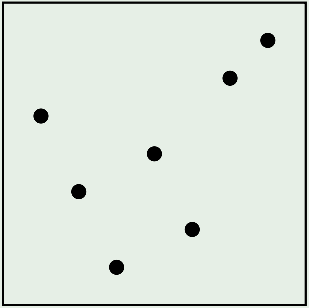
$$Av(\Pi(a, b)) \text{ is Wilf-equivalent to } Av(\Pi(c, d)).$$

- We expand on the existing proof of this result.
- We define an invertible map S, which "rotates" elements within an interval in the permutation.
- Iterating S sends permutations in $\mathrm{Av}(\Pi(a, b))$ to permutations in $\mathrm{Av}(\Pi(a-1, b+1))$

Example of the map S

Example

Consider the permutation $\pi = 5314267$, which avoids $\Pi(4,0) = \{1243, 1423, 4123\}$.
It contains $3124, 1324 \in \Pi(3,1)$.

$\pi =$  $\notin \mathrm{Av}(\Pi(3,1))$
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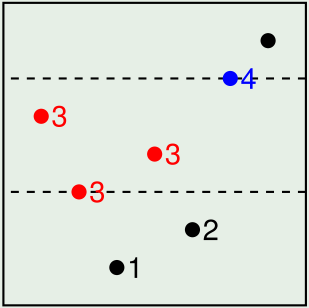
$\pi =$ 

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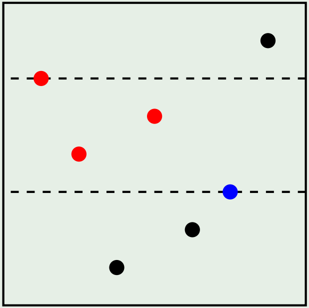

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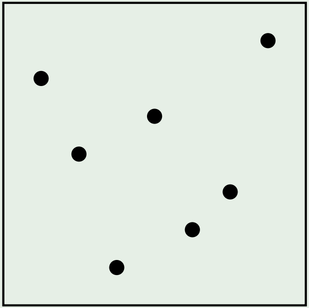

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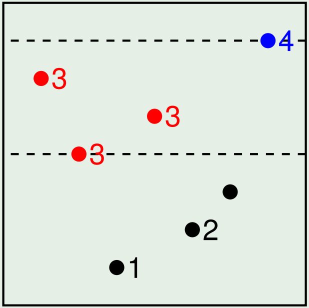

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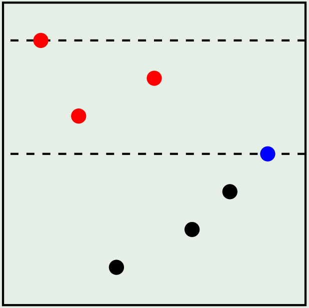
$S(S(\pi)) =$  $\in \mathrm{Av}(\Pi(3,1))$ $\notin \mathrm{Av}(\Pi(4,0))$

This map is

- Injective
- Invertible

Adding descending permutations

Theorem (Albert & Searles & SH, 2025+)

The length of the longest decreasing subpermutation in both π and $S(\pi)$ is the same.

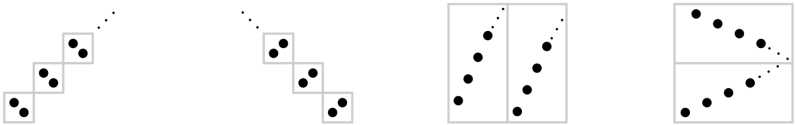
Corollary

If $a + b = c + d$, then $\Pi(a, b) \cup \delta_k$ and $\Pi(c, d) \cup \delta_k$ are Wilf-equivalent.

Peg Permutations

Theorem (Huczynska & Vatter, 2006, Albert et al, 2013.)

Avoiding arbitrarily long permutations of these forms ensures the class is enumerated by a polynomial for sufficiently large n.



Vatter and Homberger (2016) introduce *peg permutations*, a way of decorating permutations which may be inflated with arbitrarily long monotone permutations, single elements, or empty permutations. Useful to enumerate polynomial classes!

Enumerative Result

Proposition (Albert & Searles & SH 2025+)

For $a + b = \ell$ and $k \geq 1$, $\#Av_n(\Pi(a,b) \cup \delta_k)$ is a polynomial in n, with degree $(\ell - 2)(k - 2)$.

Sketch of proof:

- Consider a $+$-irreducible peg permutation representing elements of the class $\mathrm{Av}(\Pi(\ell, 0) \cup \delta_k)$.
- Can *only* arbitrarily inflate LR maxima which are above all non-LR maxima (*maximal LR maxima*).
- Non-LR maxima must avoid δ_{k-1} and also $\iota_{\ell-1}$.
- Erdős–Szekeres: at most $(k - 2)(\ell - 2)$ of these, so at most $(k - 2)(\ell - 2) + 1$ arbitrary inflations.
- Upper bound of $(k - 2)(\ell - 2)$ on degree.
- Show that bound can be achieved.

$\square$

Example

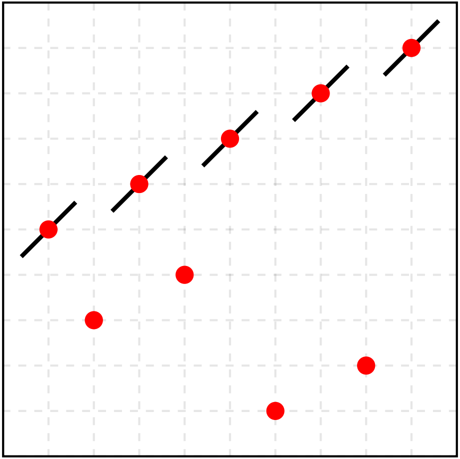


Figure: A +-irreducible peg permutation $(5^{+}3^{\bullet}6^{+}4^{\bullet}7^{+}1^{\bullet}8^{+}2^{\bullet}9^{+})$ which lies in $\mathrm{Av}(\Pi(4,0) \cup \delta_4) = \mathrm{Av}(1243, 1423, 4123, 4321)$.

Interlude

Lemma

The number of sequences $a_1 a_2 \ldots a_{k-1} a_k$ of length k such that for all $j \leq k$,

$$\sum_{i \leq j} a_i < j$$

is given by C_k, the k^{th} Catalan number.

Example

The sequences of length 3 are 000,001,010,011, and 002. ($C_3 = 5$).

Catalan leading term

Proposition (Albert & Searles & SH 2025+)

For $\ell = a + b \geq 3$:

$$\#Av_n(\Pi(a,b) \cup \delta_3) \textit{ has leading term } C_{\ell-2}\binom{n}{\ell-2}.$$

Sketch of proof.

- From previous theorem, leading term is $\alpha\binom{n}{\ell-2}$.
- Count peg permutations as in previous theorem, which contribute to maximal degree term (i.e. have $(\ell-2)(k-2)+1 := \ell-1$ maximal LR maxima.
- Separated by $\ell-2$ increasing elements.
- Number of ways to place *non-maximal* LR maxima gives the sequence from the previous lemma.

$\square$

Example

Let $\ell = 5$. Then, consider the ways to create a peg permutation for the class $\mathrm{Av}(\Pi(5,0) \cup \delta_3)$.

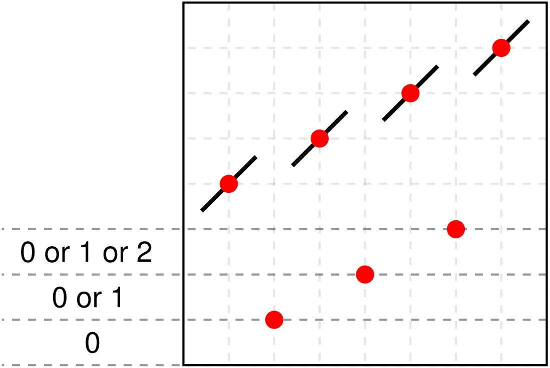


Figure: Generic element in
$\mathrm{Av}(\Pi(5,0) \cup \delta_3) = \mathrm{Av}(12354, 12534, 15234, 51234, 321)$.

A conjectured closed form

Conjecture (Albert & Searles & SH 2025+)

For $\ell = a + b \geq 3$ and $n > \ell$,

$$\#Av_n(\Pi(a,b) \cup \delta_3) = C_{\ell-2}\binom{n}{\ell-2} - \sum_{1 \leq h < n-2} T_{\ell-2,h}\binom{n}{\ell-3-h},$$

where $T_{p,q} = \frac{q\binom{2p-q}{p}}{2p-q}$ (OEIS sequence A033184, Transposed Catalan Triangle).

What about riffle-shuffles?

Definition

Let $\rho \in \mathfrak{S}_a$ and $\sigma \in \mathfrak{S}_b$. Then $\rho \sqcup\!\sqcup\, \sigma$ is the set of permutations $\pi \in \mathfrak{S}_{a+b}$ such that π consists of two subpermutations, which are order isomorphic to ρ and σ.

Example

Shuffles of 123 with 21:

12354, 12534, 15234, 51234, 12543,
15243, 51243, 15423, 51423, 54123.

Definition

We define the shuffle of two avoidance class bases Π_1 and Π_2 to be

$$\Pi_1 \sqcup\!\sqcup\, \Pi_2 = \{\pi \sqcup\!\sqcup\, \sigma : \pi \in \Pi_1, \sigma \in \Pi_2\}$$

Riffle-shuffles of avoidance class bases

Theorem (Albert & Searles & SH 2025+)

Let $A(x)$ and $B(x)$ be the exponential generating functions that count elements of $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ respectively. Then, the exponential generating function for $\mathrm{Av}(C) := \mathrm{Av}(A \sqcup B)$ is

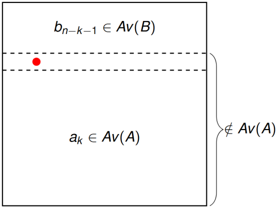
$$A(x) + B(x) + (x-1)A(x)B(x).$$

Example

$\mathrm{Av}(12)$ has exponential generating function e^x and $\mathrm{Av}(123, 132)$ has e.g.f $\frac{e^{2x}+1}{2}$. so

$$\mathrm{Av}(12 \sqcup \{123, 132\}) = \mathrm{e}^{\mathrm{x}} + \frac{\mathrm{e}^{2\mathrm{x}} + 1}{2} + \frac{(\mathrm{x}-1)(\mathrm{e}^{3\mathrm{x}} + \mathrm{e}^{\mathrm{x}})}{2}.$$

Sketch of proof



- $\mathrm{Av}(A), \mathrm{Av}(B) \subset \mathrm{Av}(A \sqcup B)$.
- If smallest k elements in $\mathrm{Av}(A)$, add an element to get something not in $\mathrm{Av}(A)$ in $(k+1)a_k - a_{k+1}$ ways.
- Above this, b_{n-k-1} subpermutations, position these in $\binom{n}{n-k-1} = \binom{n}{k+1}$ ways.
- Summing over k gives

$$c_n = a_n + \sum_{k=0}^{n-1} \big((k+1)a_k - a_{k+1}\big) b_{n-k-1} \binom{n}{k+1}.$$

- Divide through by $n!$, distribute denominator of binomial term:

$$\frac{c_n}{n!} = \frac{a_n}{n!} + \sum_{k=0}^{n-1} \left(\frac{a_k}{k!} - \frac{a_{k+1}}{(k+1)!} \right) \frac{b_{n-k-1}}{(n-k-1)!}.$$

Thanks and acknowledgments

A big thank you to the organisers. Attending this conference was made possible with funding from the New Zealand Mathematical Society and the University of Otago.

Questions?