Joint Packing Densities and the Great Limit Shape

Permutation Patterns
St Andrews, July 10, 2025

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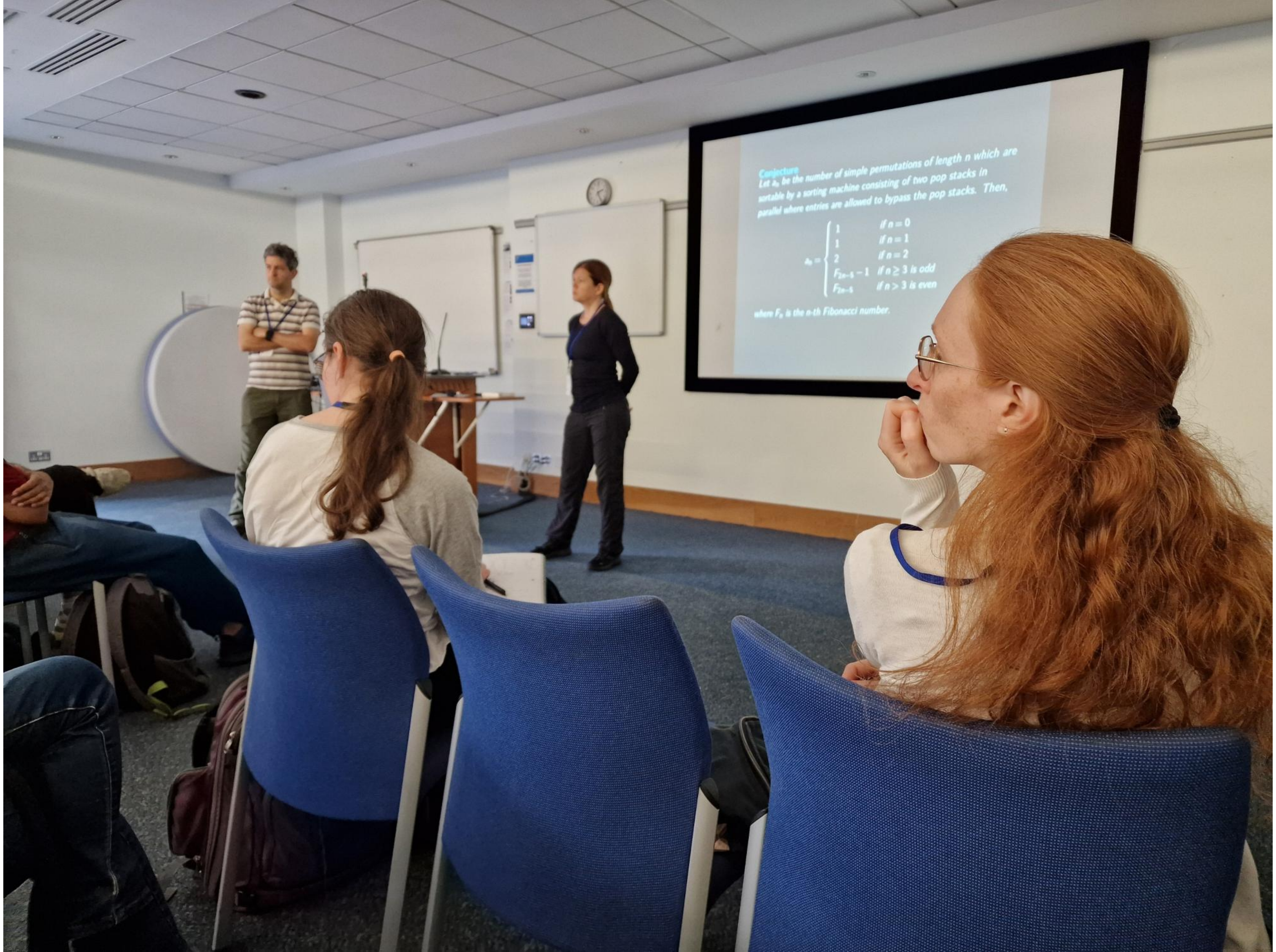
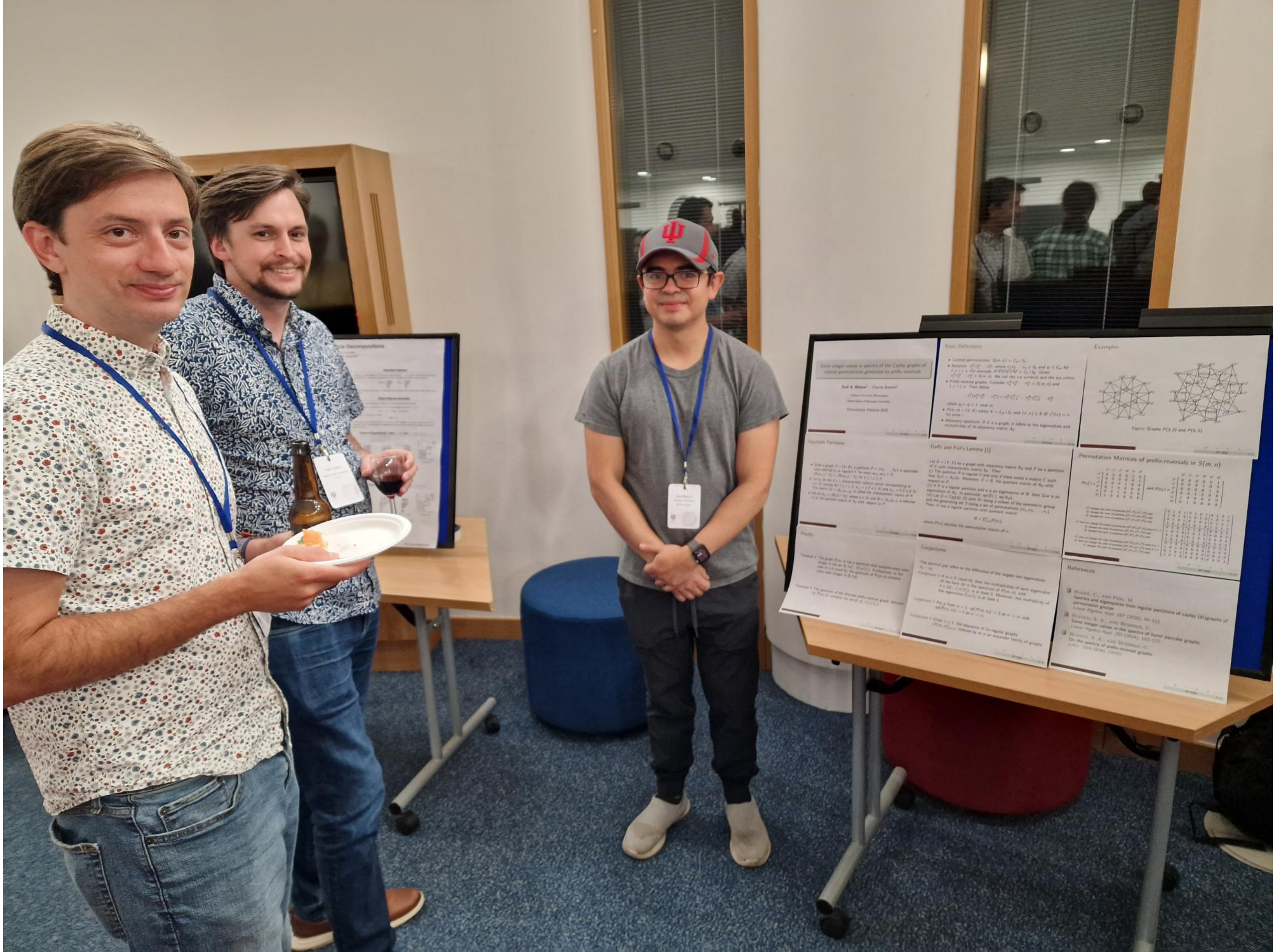
Schemes for Pattern-Avoiding Words


MSU

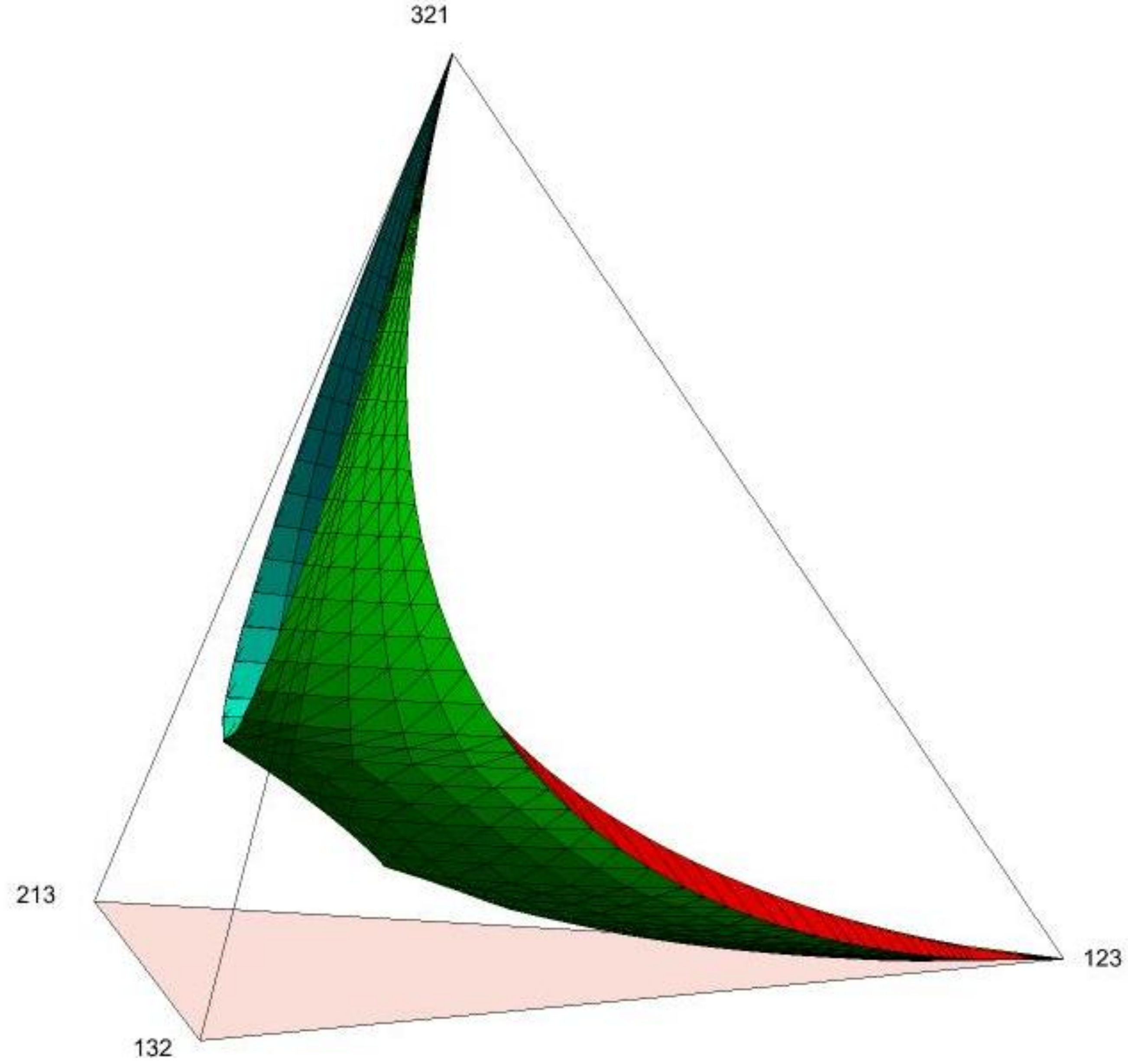

Conjecture
Let a_n be the number of simple permutations of length n which are sortable by a sorting machine consisting of two pop stacks in parallel where entries are allowed to bypass the pop stacks. Then,
$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ F_{2n-5} - 1 & \text{if } n \geq 3 \text{ is odd} \\ F_{2n-5} & \text{if } n > 3 \text{ is even} \end{cases}$$
where F_n is the n-th Fibonacci number.



Basic Definitions
Examples
Figure: Graphs P(3,2) and P(4,2)
Equitable Partitions
Dalfó and Fiol's Lemma [1]
Permutation Matrices of prefix-reversals in S(m, n)
Results
Conjectures
References


Walking
Tours In...
Scotland

Great Limit Shape (layered version)



Joint Work

This talk includes joint work by

Sergi Elizalde,

Mark Noy,

Anna de Mier.

Shoutouts: Miles Jones

Lara Pudwell

Another Source: Permutations with fixed pattern densities
Richard Kenyon, Daniel Král', Charles Radin, Peter Winkler
Random Structures & Algorithms 56(1) Jan. 2020

Packing Density

Pick a pattern, say σ = 132. The *packing density* of 132 in a permutation π is

$$\delta_{132}\ (\pi)\ =\ \text{fraction of 3-term subsequences in } \pi \text{ that have order type } 132.$$

What values can occur?

Packing Density

Pick a pattern, say σ = 132. The *packing density* of 132 in a permutation π is

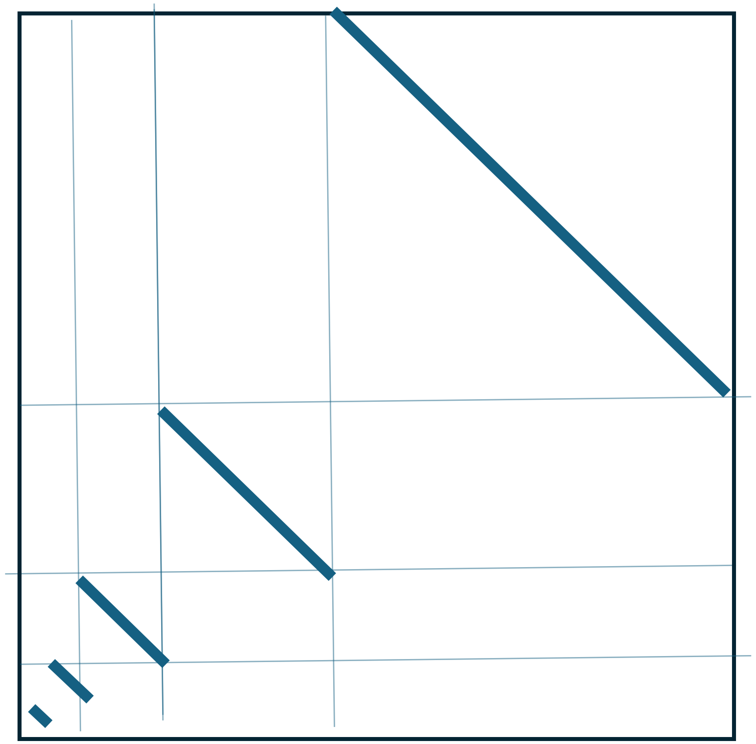
$\delta_{132}\ (\pi)\ =$ fraction of 3-term subsequences in π
that have order type 132.

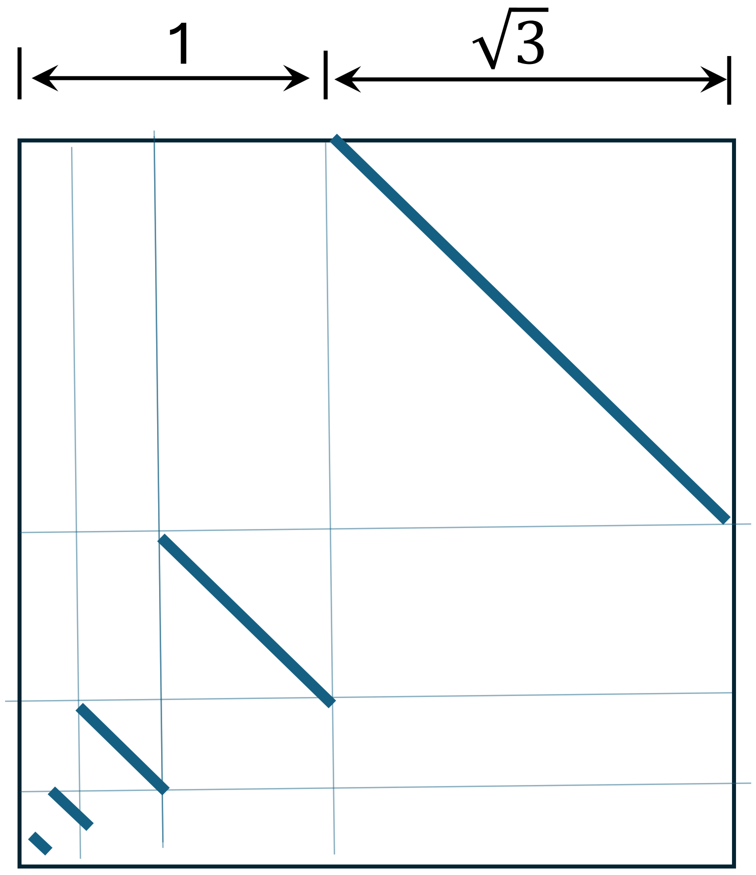
What values can occur?

We care about the limit of large π, so we want to know what values x can occur as

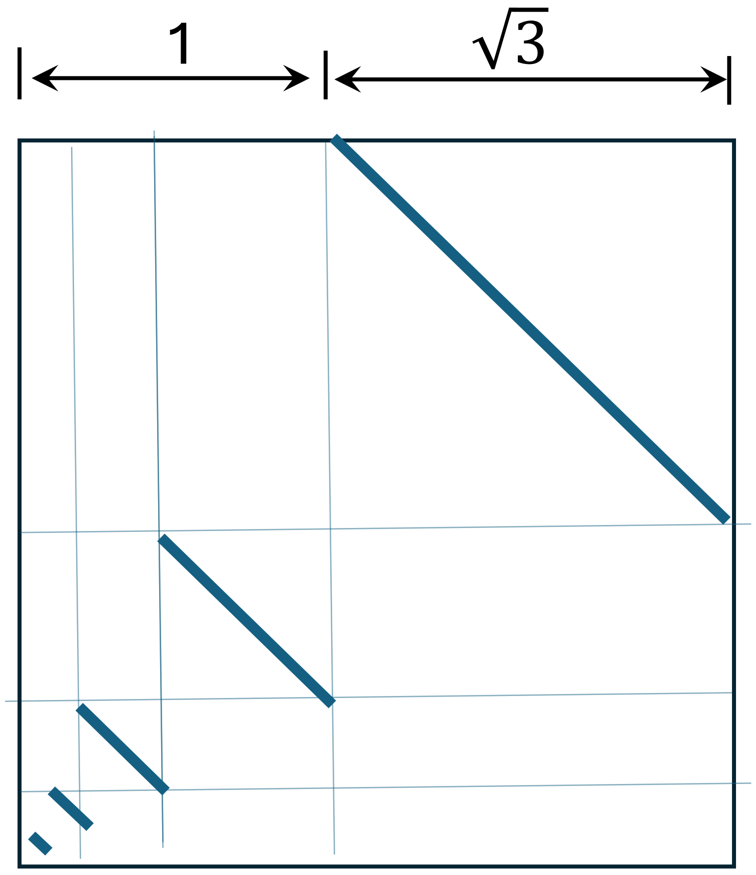
$$\lim_{i\to\infty} \delta_{132}(\pi_i)$$

for a sequence of permutations $\pi_1, \pi_2, \ldots$.

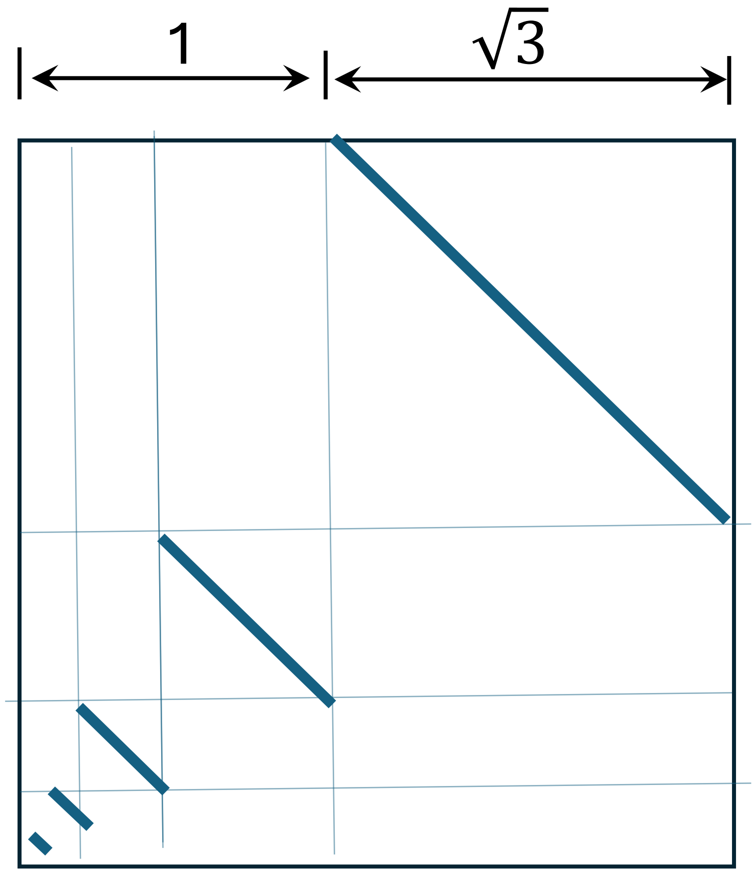




1
√3



$$\delta_{132}(\pi) = 2\sqrt{3} - 3 \approx 0.464\ldots$$



$$\delta_{132}(\pi) = 2\sqrt{3} - 3 \approx 0.464$$

$$\delta_{321}(\pi) = \frac{3}{2} - \sqrt{3} \approx 0.268$$

Joint Packing Densities

Which vectors

$$v = (\delta_{321}, \delta_{132}) \in R^2$$

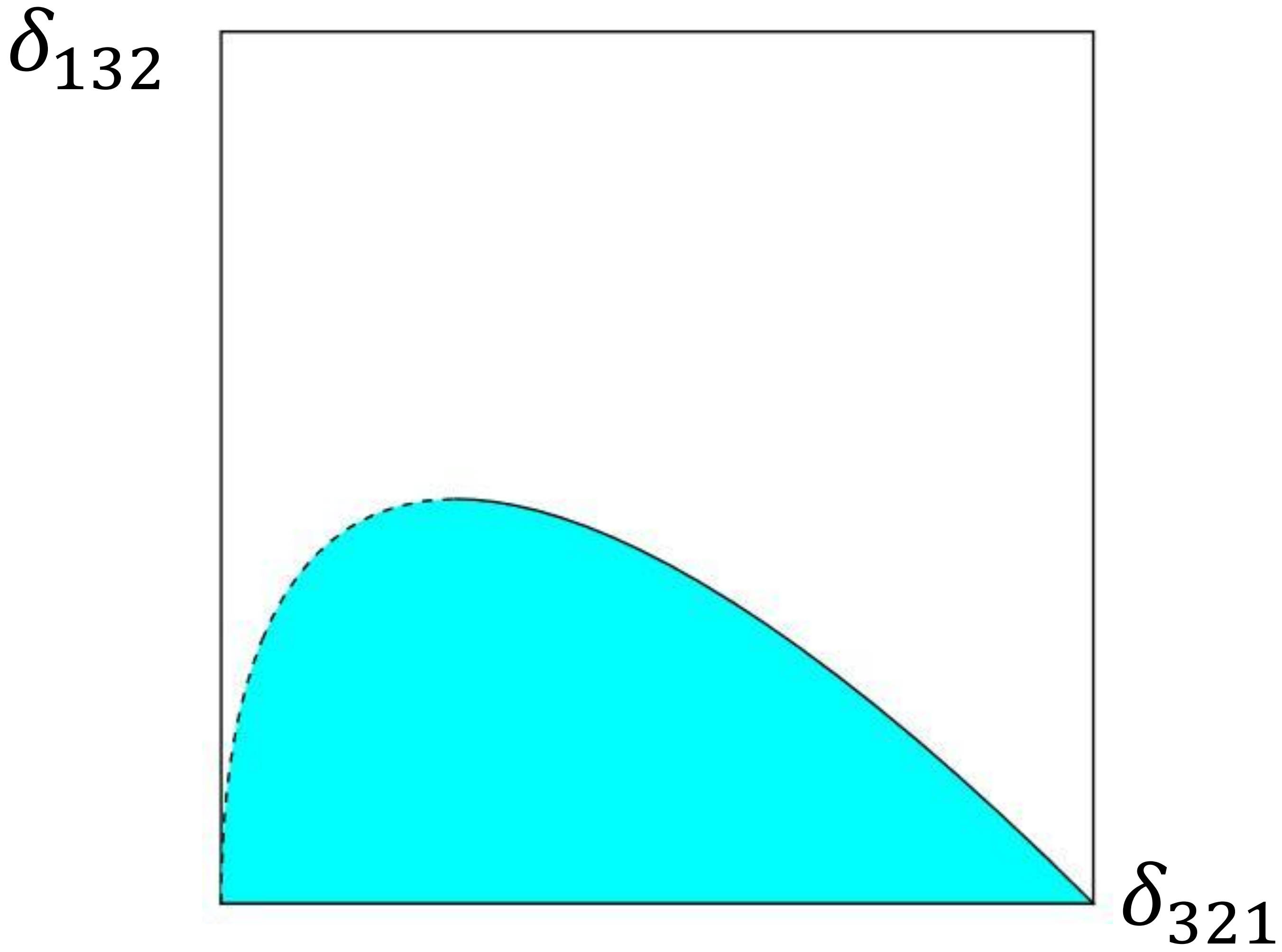
can occur as a limit of vectors

$$(\delta_{321}(\pi_i), \delta_{132}(\pi_i))$$

for a sequence of permutations $\pi_1, \pi_2, \pi_3, \ldots$ of increasing size?

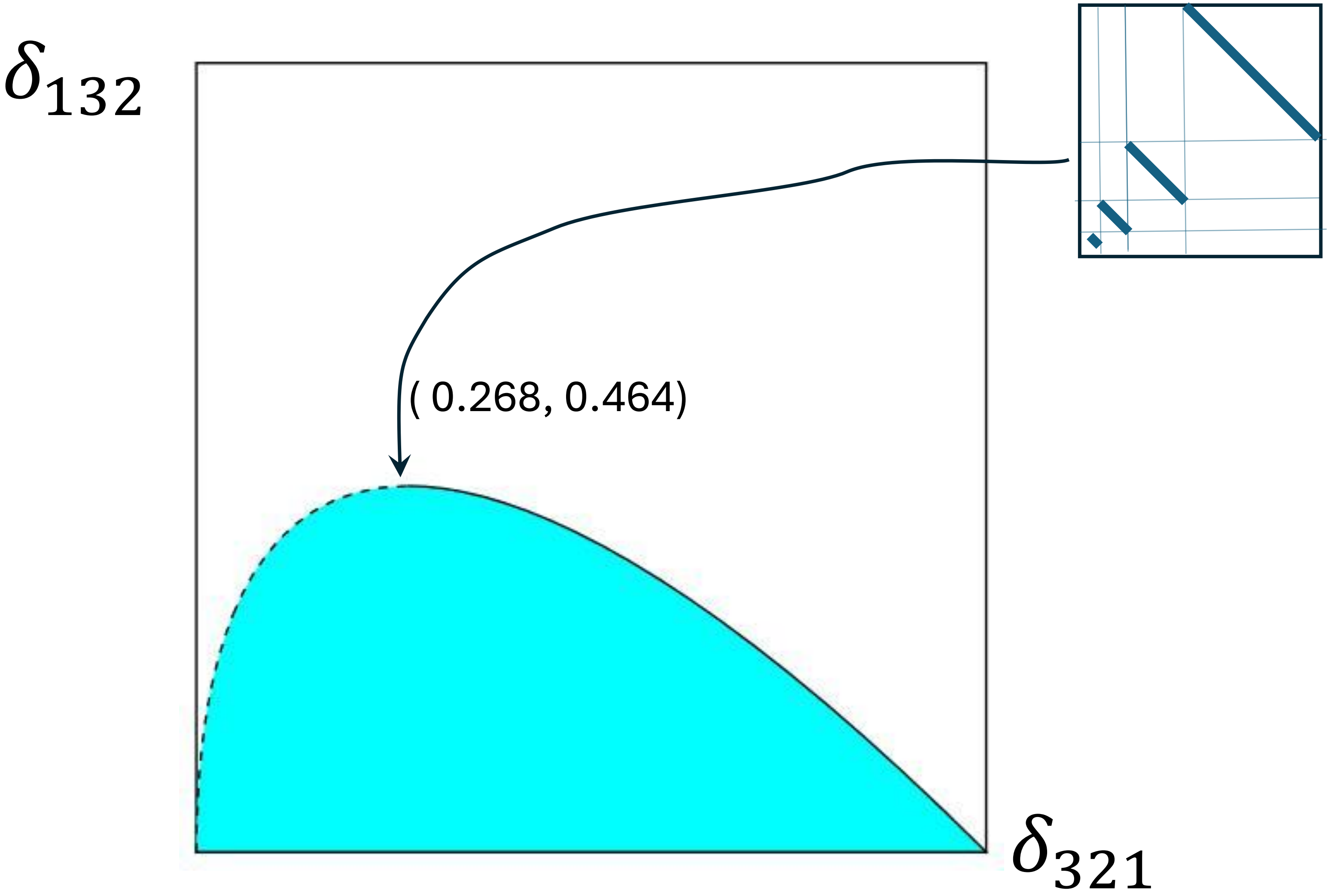
Call the answer Π (321, 132). It is a compact subset of R^2.

Joint Packing Density for δ_{321} and δ_{132}

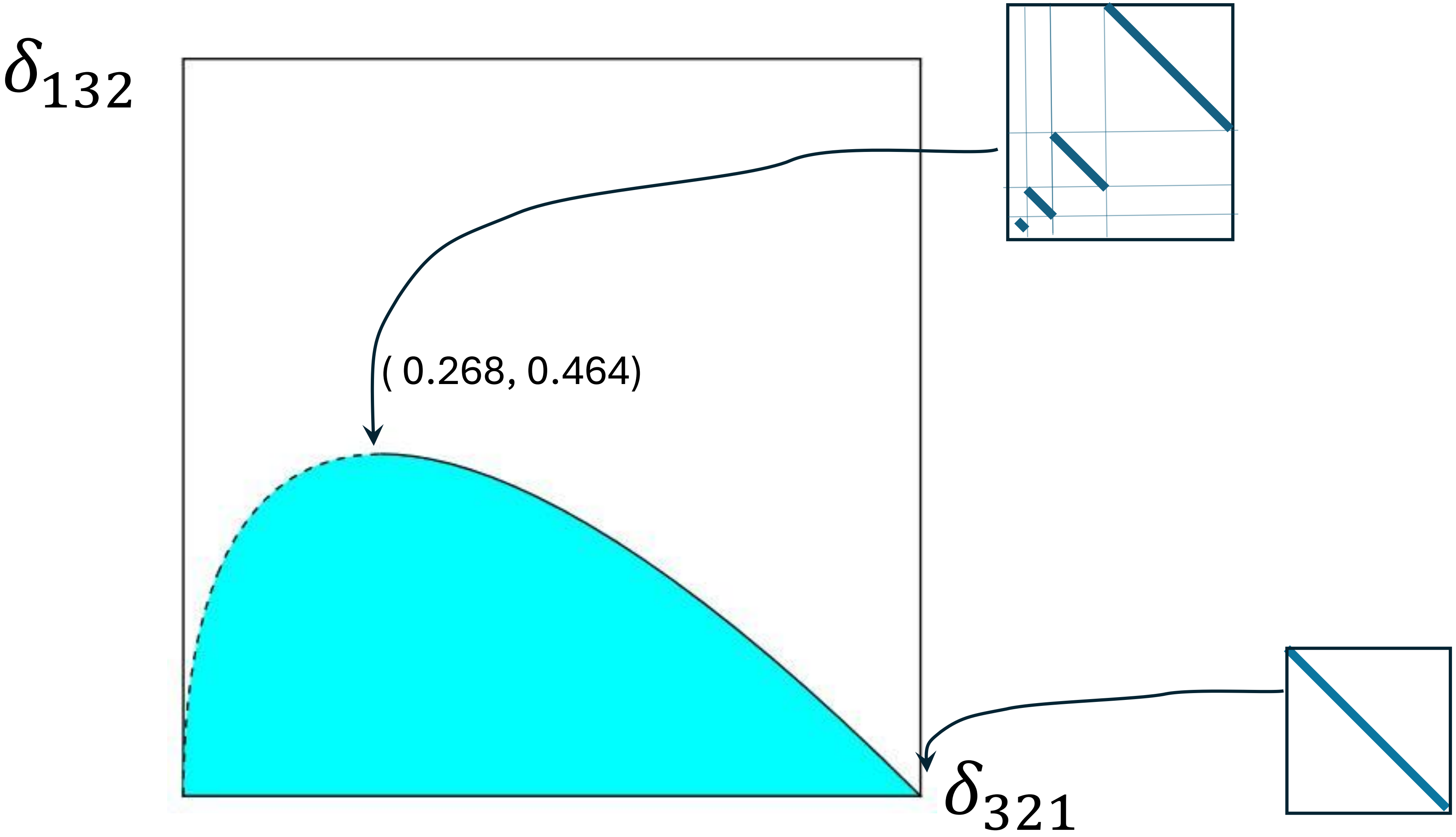


Possible limiting values of the pair (δ_{321} , δ_{132})

Joint Packing Density for δ_{321} and δ_{132}



Joint Packing Density for δ_{321} and δ_{132}



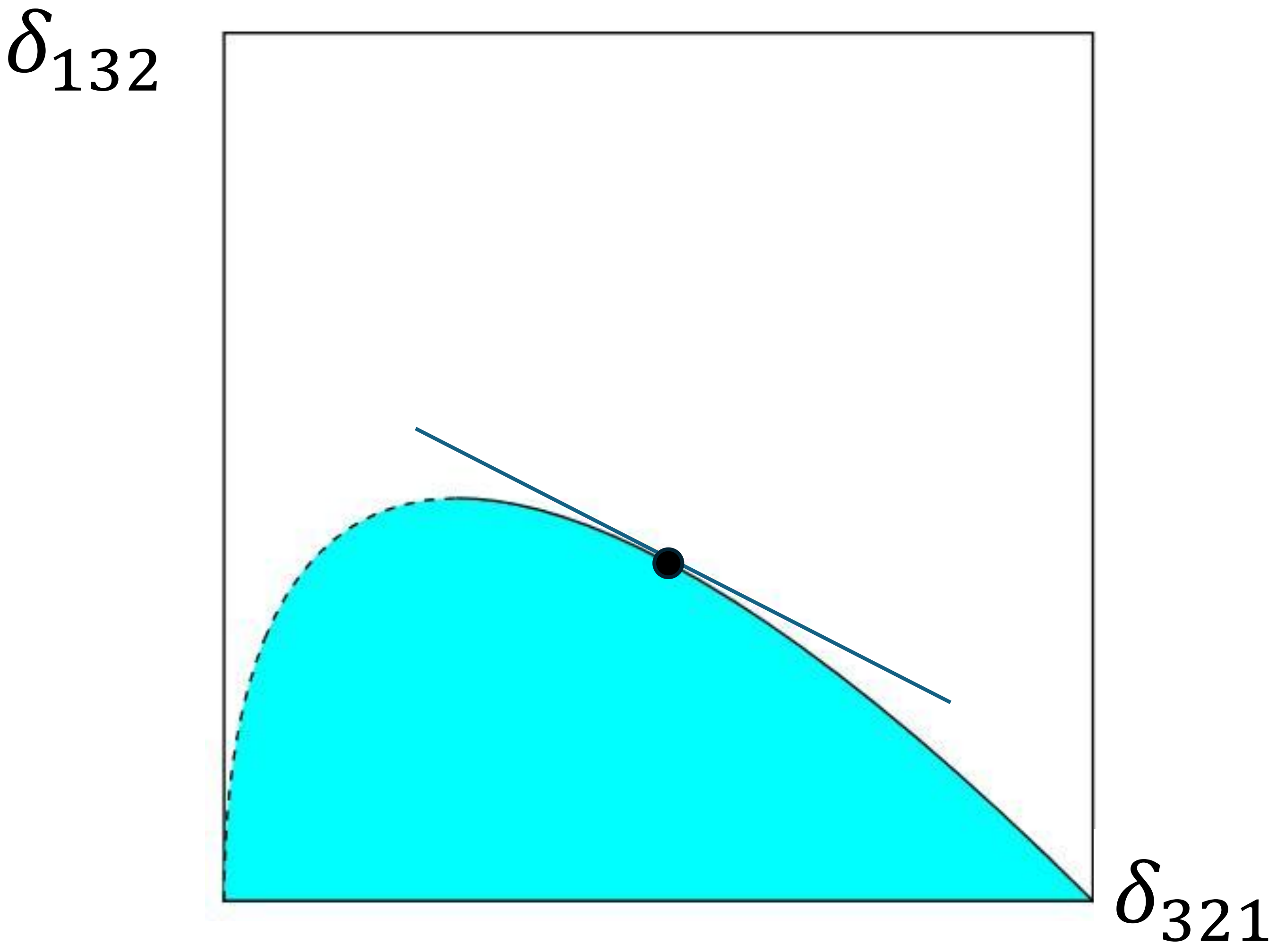
Joint Packing Density for δ_{321} and δ_{132}

Every point on this boundary maximizes a function of the form

$$\delta_{132} + k\,\delta_{321}$$

On the downward-sloping part of the curve, k is non-negative.

The permutons that maximize such a function are layered permutons.



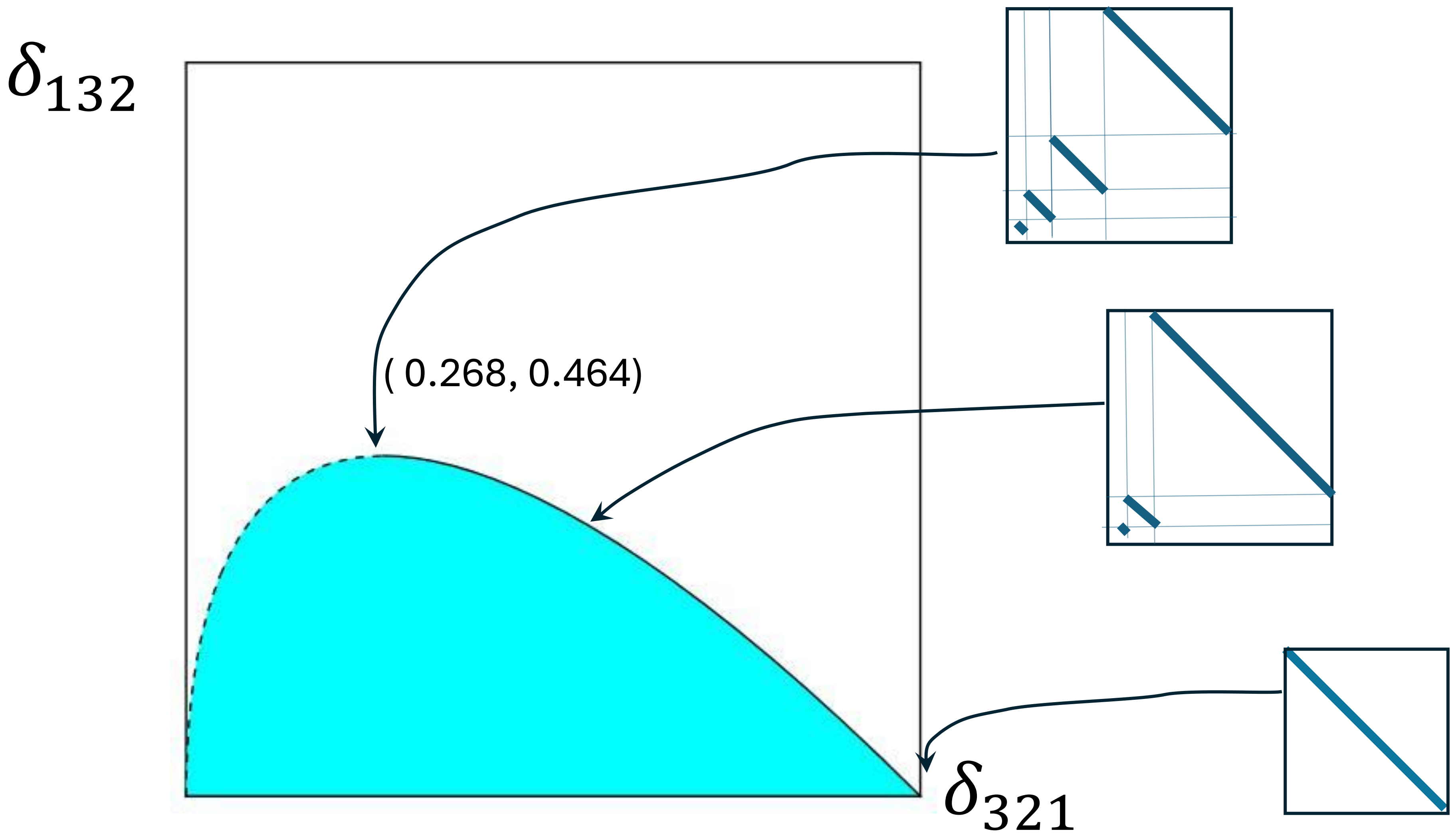
Theorem: Layered patterns have layered optimizers.

Theorem: Linear combinations of layered patterns

(like $\delta_{132} + k\,\delta_{321}$)

have layered optimizers…IF the coefficients are non-negative.

Joint Packing Density for δ_{321} and δ_{132}



Theorem: Layered patterns have layered optimizers.

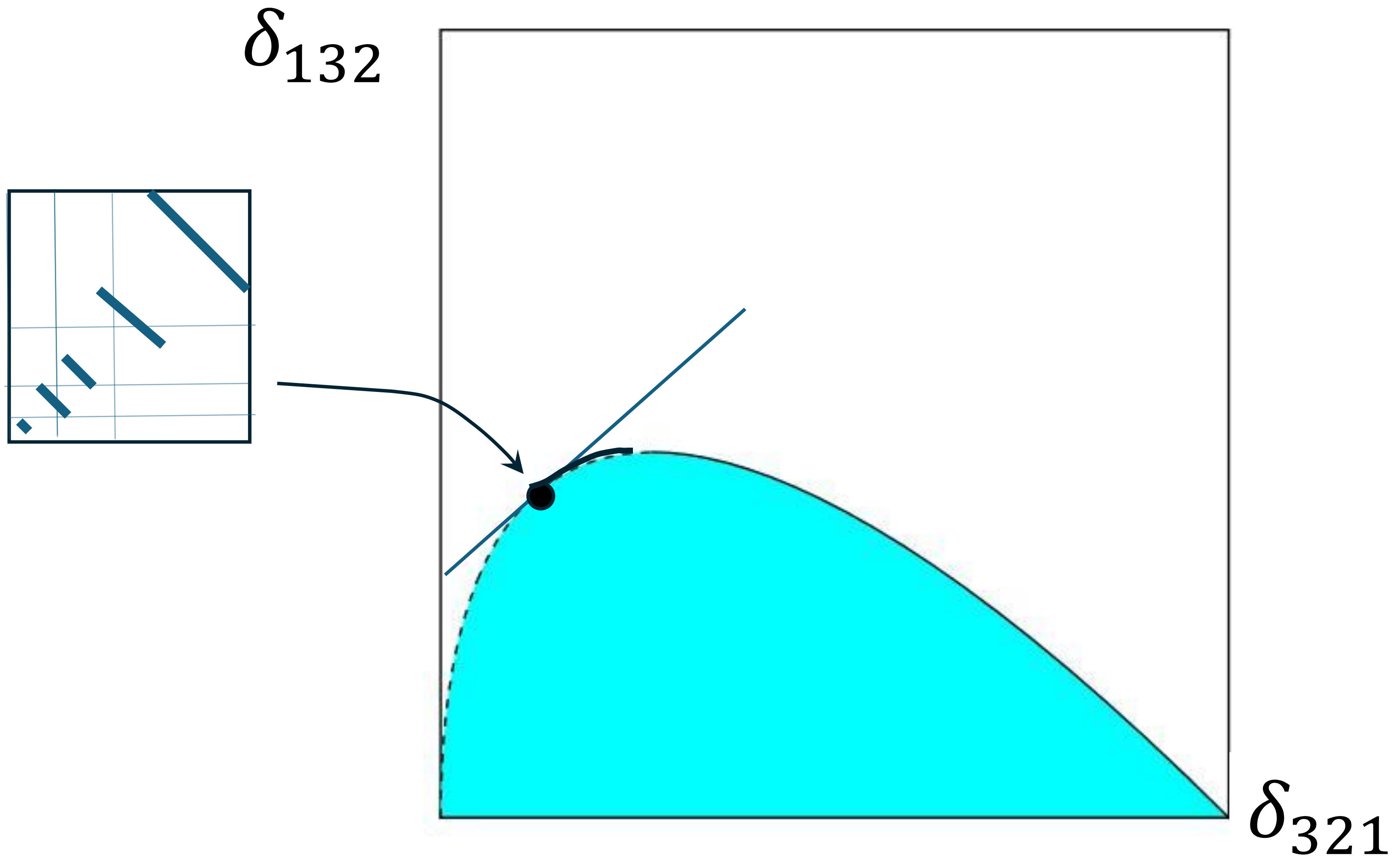
Theorem: Linear combinations of layered patterns

(like $\delta_{132} + k\, \delta_{321}$)

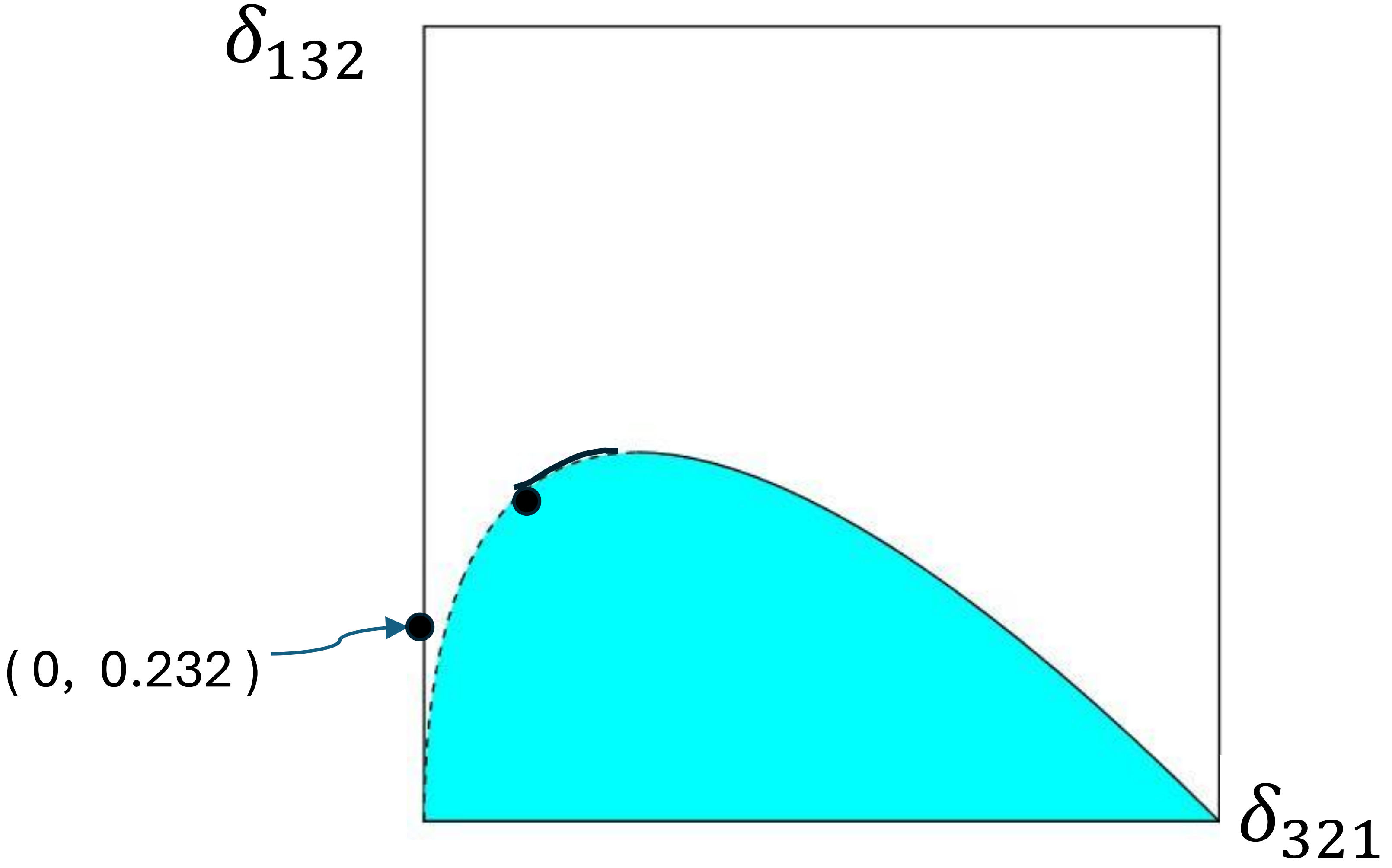
have layered optimizers…IF the coefficients are non-negative.

Theorem: The function $\delta_{132} + k\, \delta_{321}$ has a layered optimizer if $k \geq -1$.

Joint Packing Density for δ_{321} and δ_{132}



Joint Packing Density for δ_{321} and δ_{132}



Joint Packing Density for δ_{321} and δ_{132}

Lara Pudwell looked at restricted packing densities.

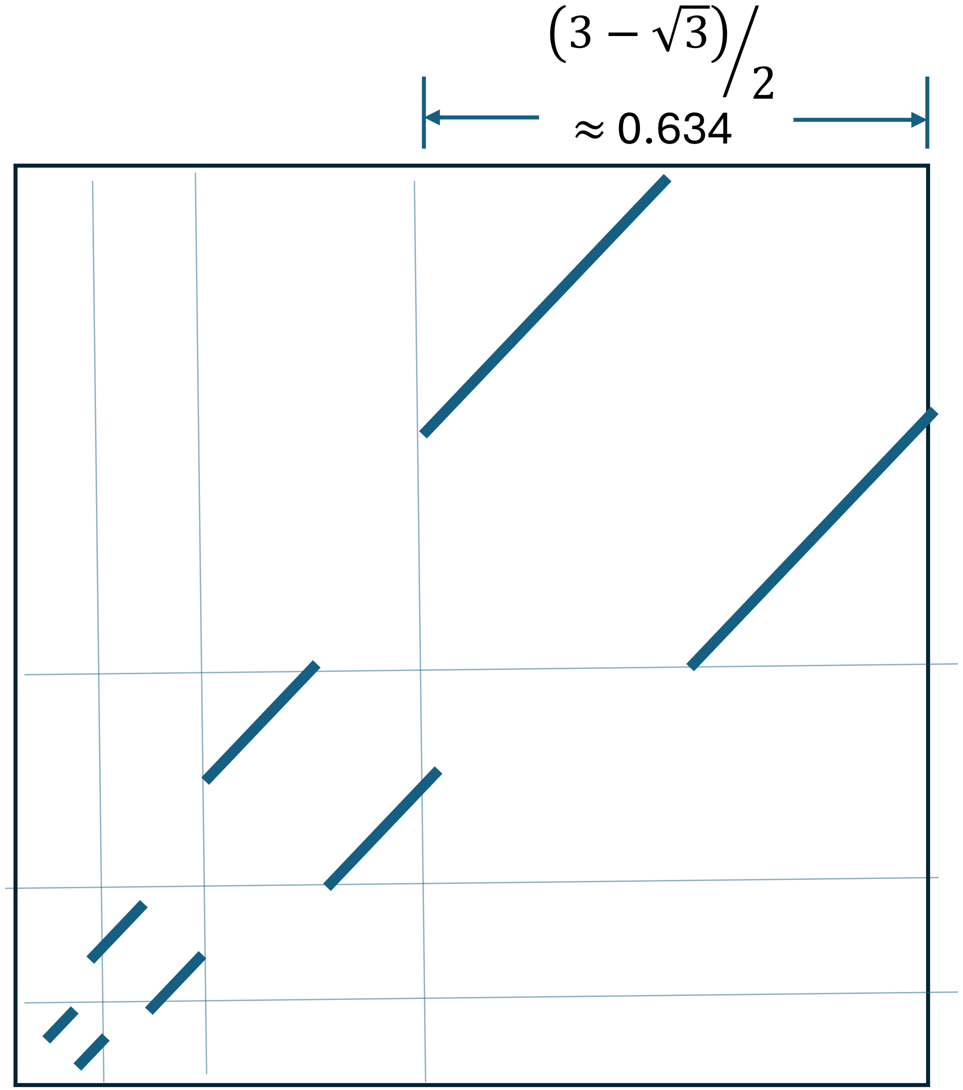
For permutations avoiding 321, the packing density of 132 is

$$\frac{3}{2} - \sqrt{3} \approx 0.232.$$

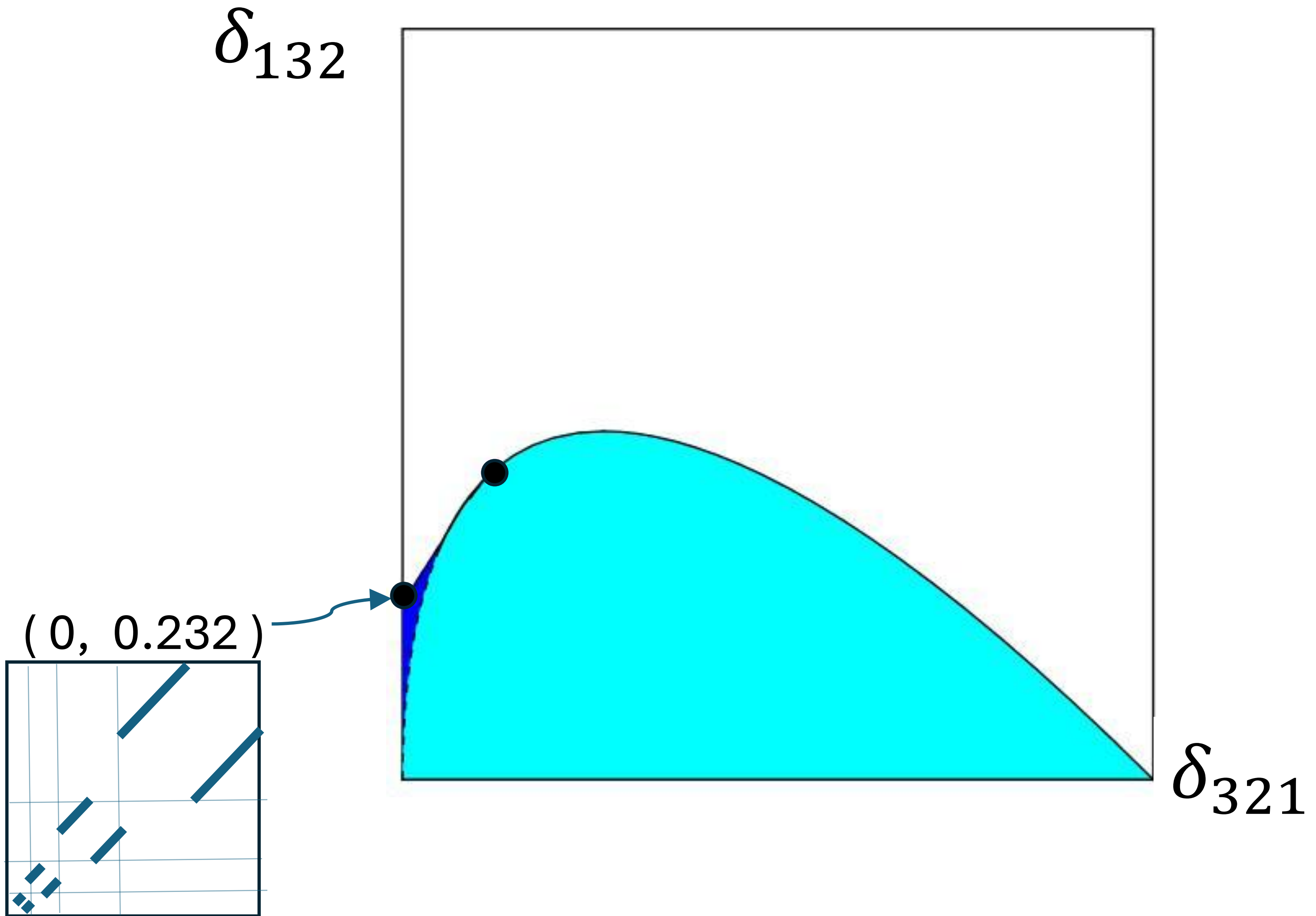
That is,

$(\delta_{321}, \delta_{132})$ can be

(0, 0.232).



Joint Packing Density for δ_{321} and δ_{132}



The Great Limit Shape

Which vectors

$$v = (\delta_{123}, \delta_{132}, \delta_{213}, \delta_{231}, \delta_{312}, \delta_{321}) \in R^6$$

can occur as a limit of vectors

$$(\delta_{123}(\pi_i), \delta_{132}(\pi_i), \delta_{213}(\pi_i), \delta_{231}(\pi_i), \delta_{312}(\pi_i), \delta_{321}(\pi_i))$$

for a sequence of permutations $\pi_1, \pi_2, \pi_3, \ldots$ of increasing size?

The answer is a compact subset of R^6.

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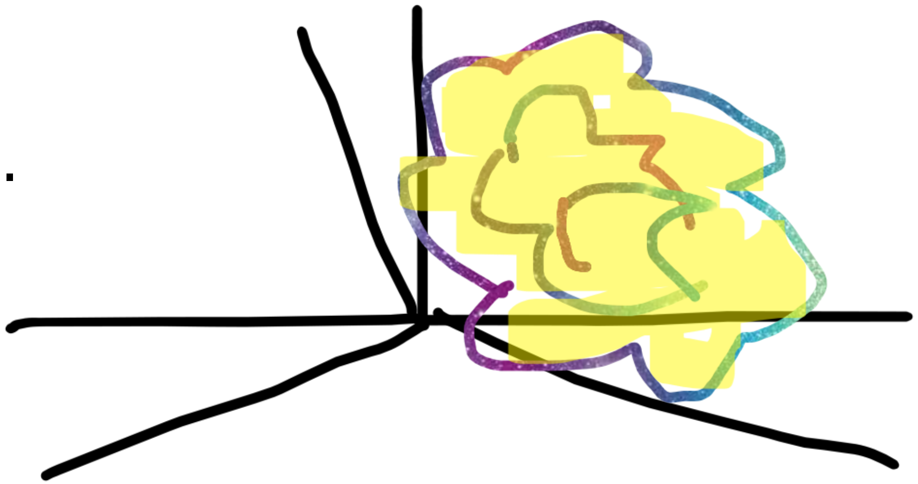
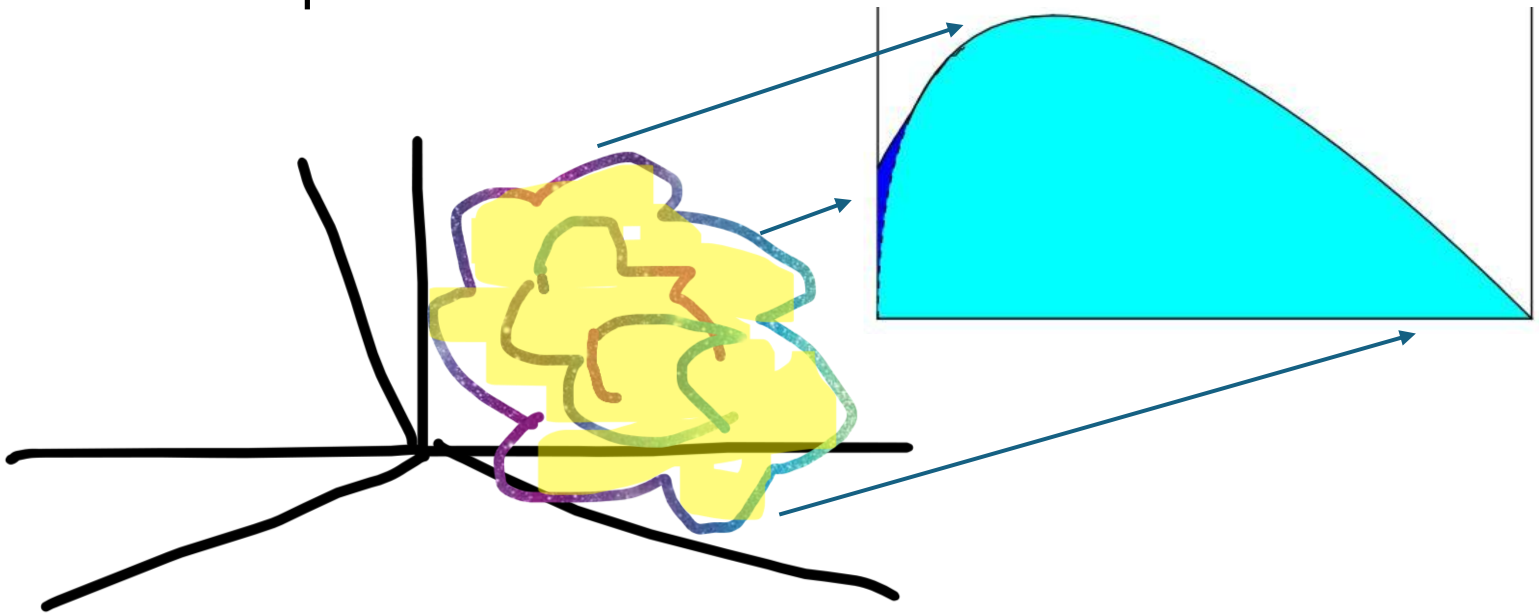


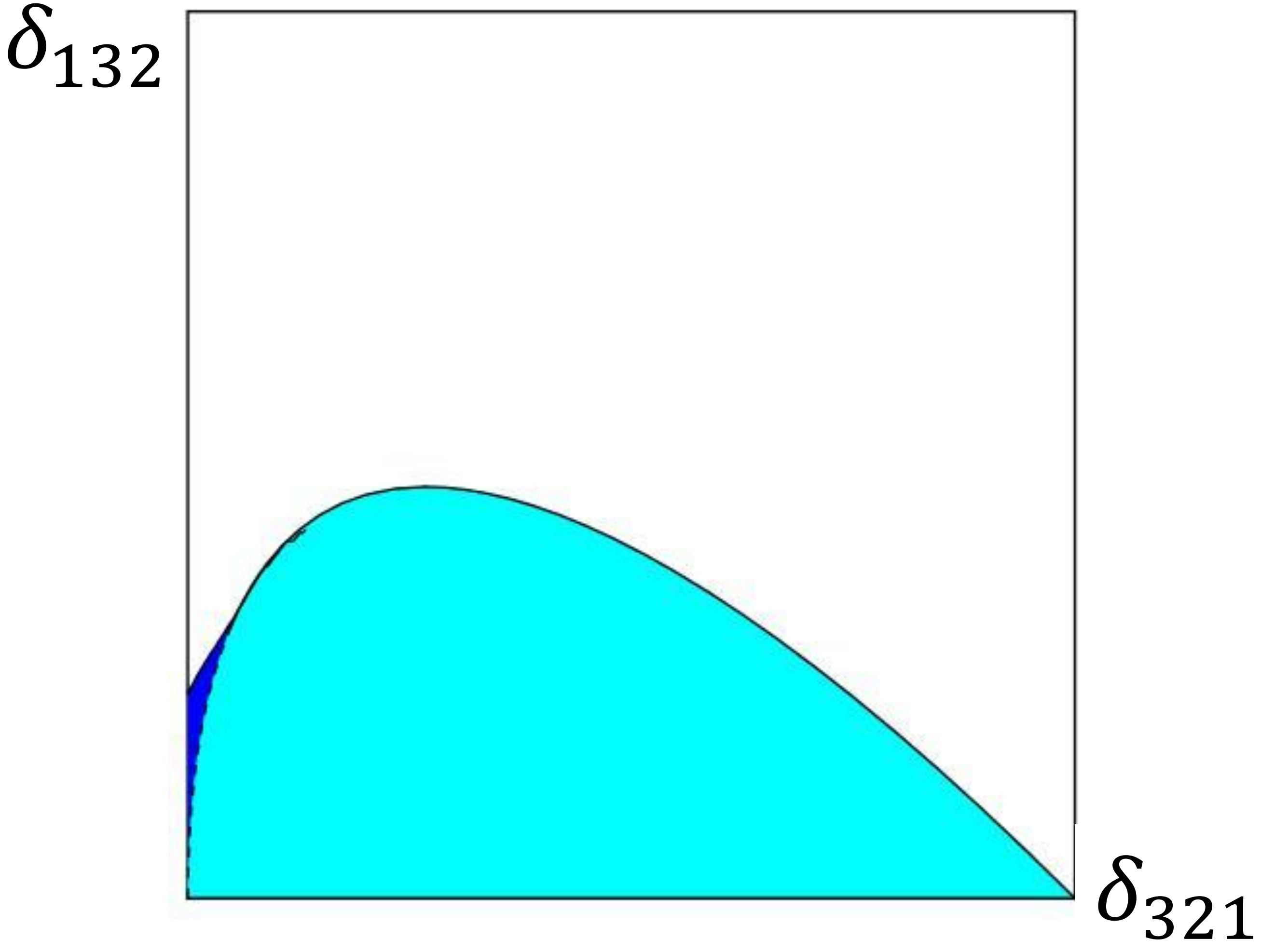
The Great Limit Shape

We can understand the Great Limit Set in terms of its PROJECTIONS and CROSS SECTIONS.

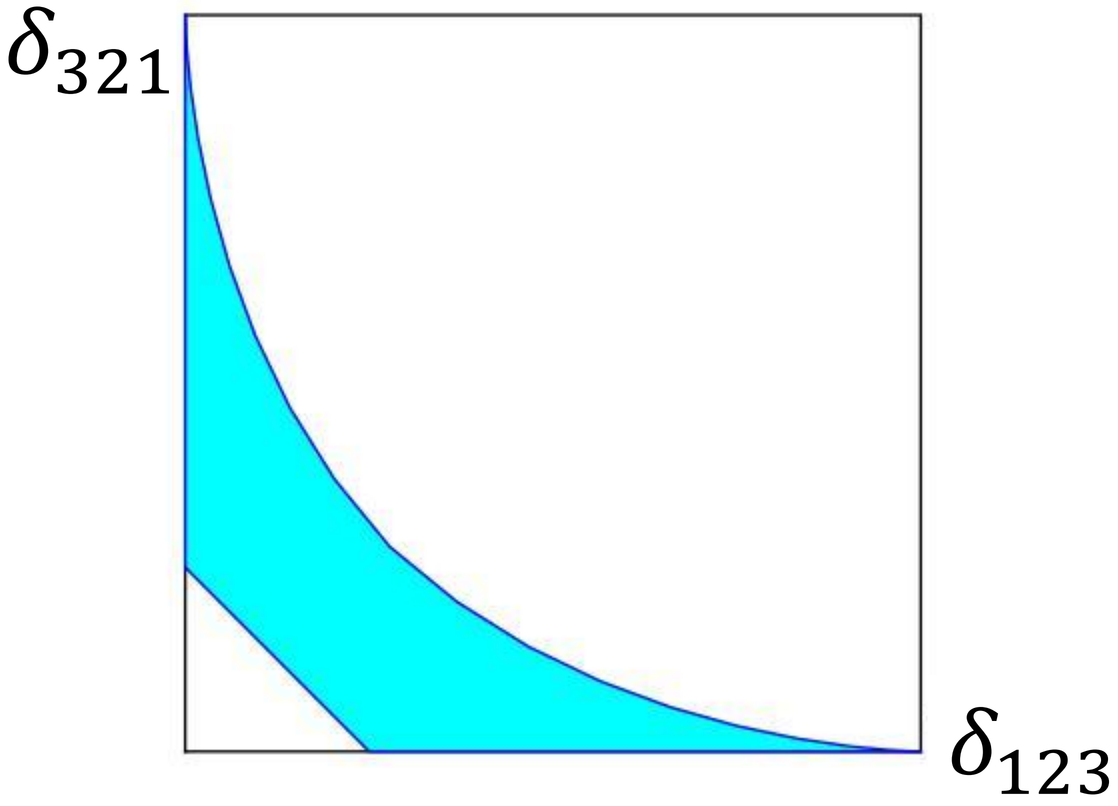
The solution we saw to the $(\delta_{123},\ \delta_{321})$ problem is an example of a projection of the 5-d set onto a 2-d plane.



Joint Packing Density for δ_{321} and δ_{132}



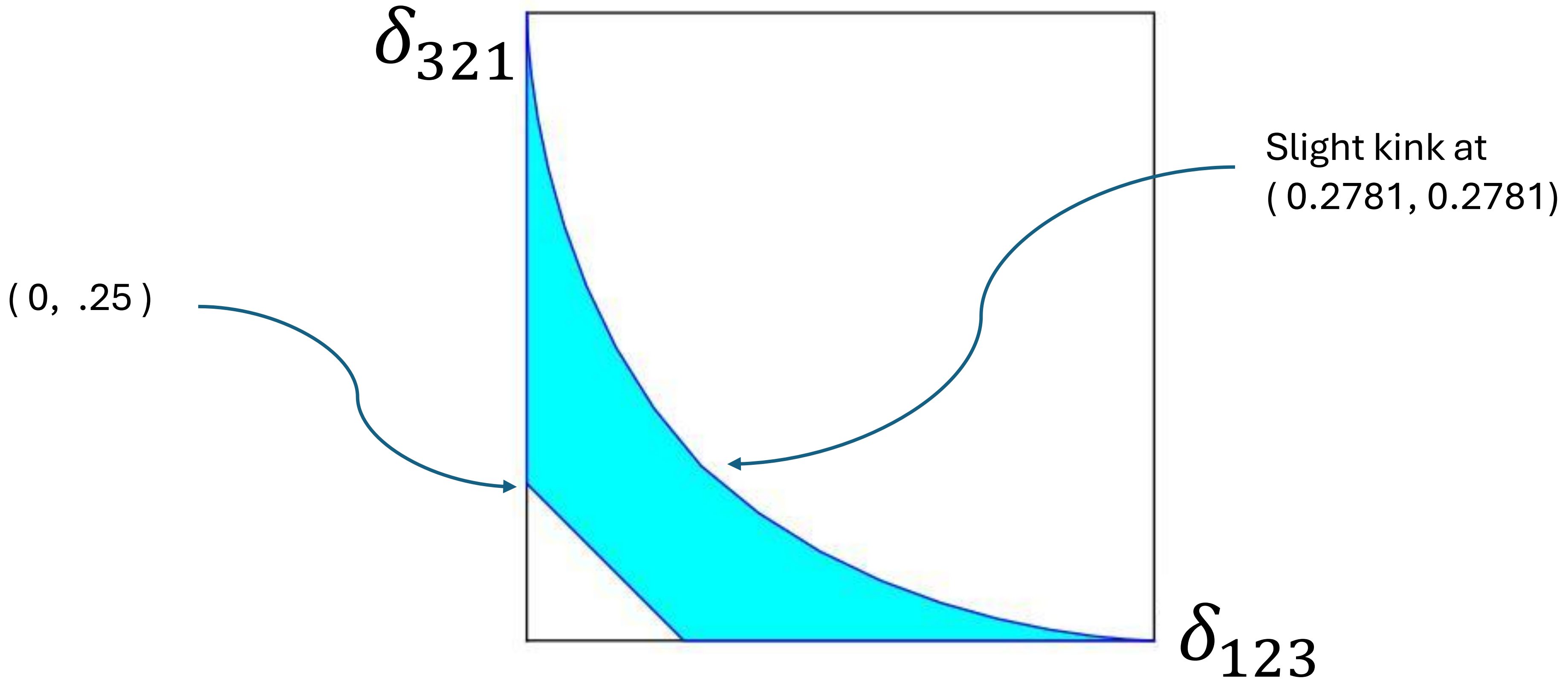
Joint Packing Density for δ_{123} and δ_{321}



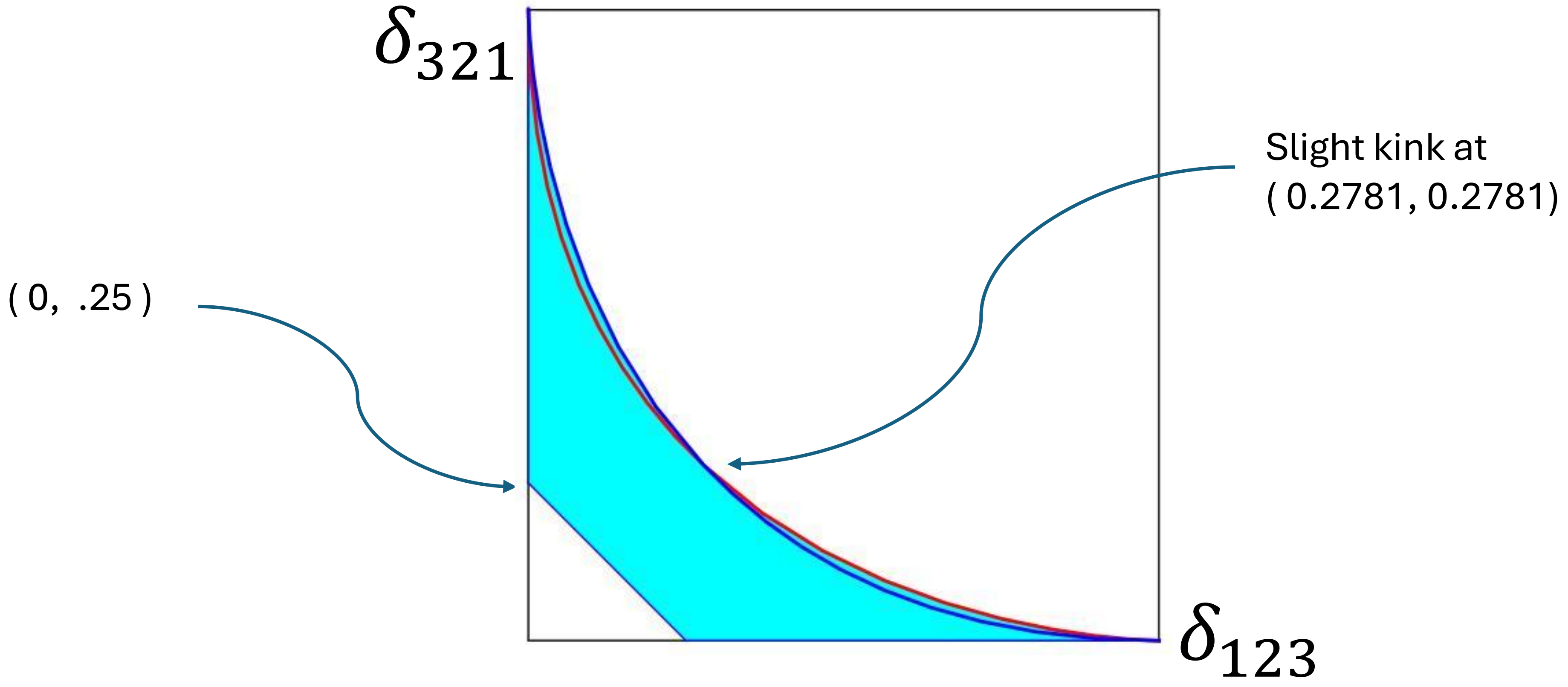
Possible limiting values of the pair (δ_{123}, δ_{321})

This diagram and proof of its correctness are due to Sergi Elizalde and his collaborators.

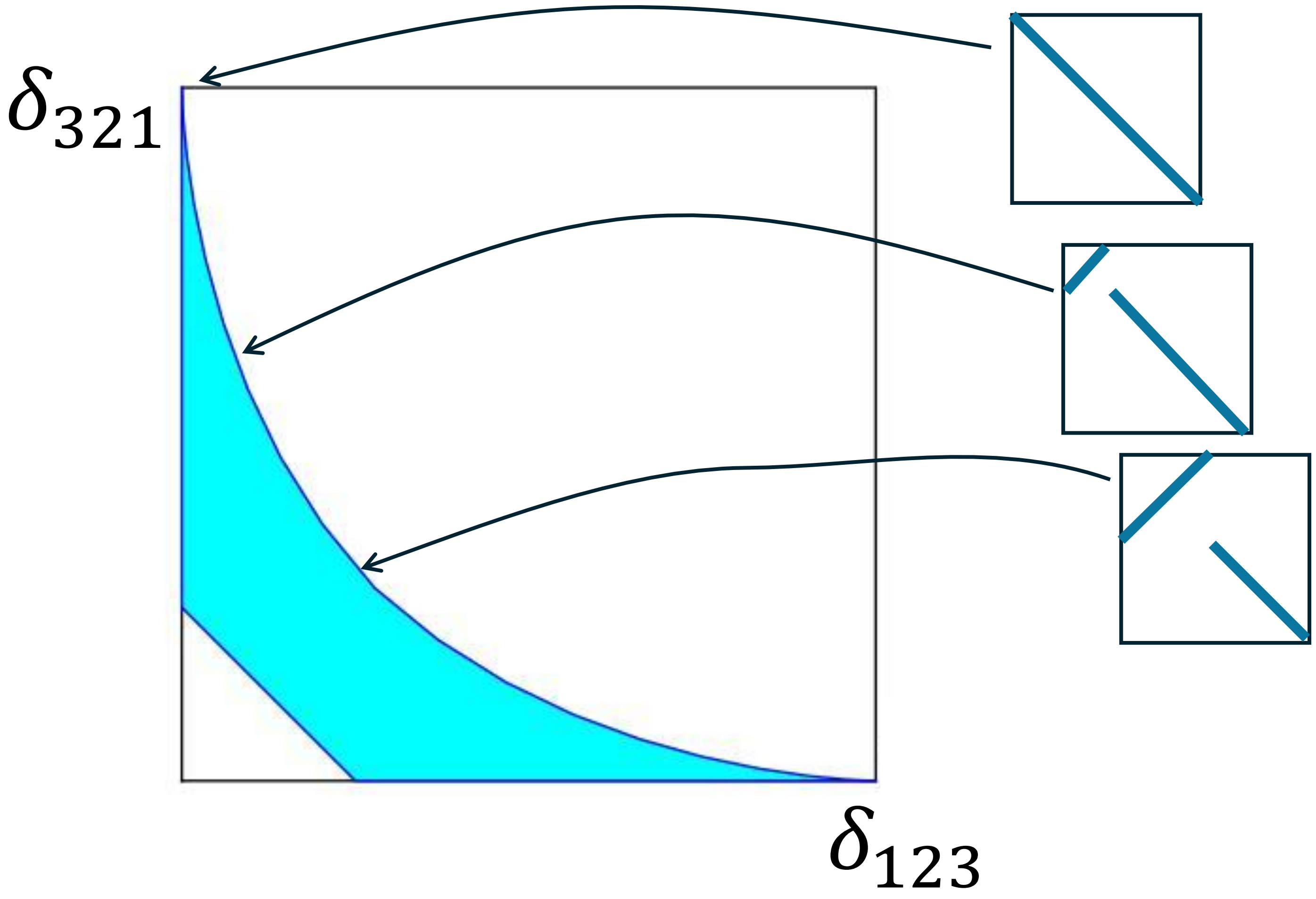
Joint Packing Density for δ_{123} and δ_{321}



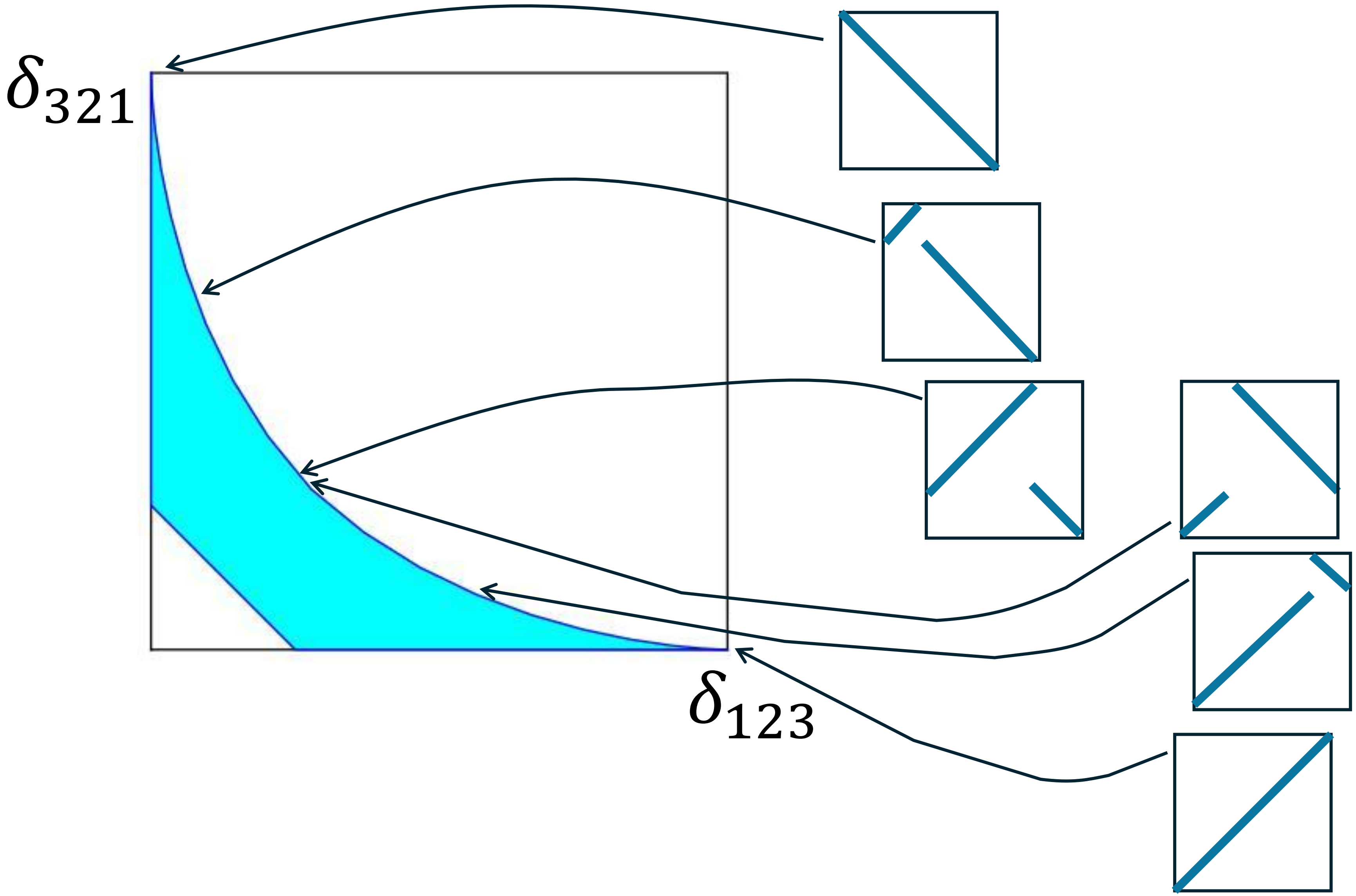
Joint Packing Density for δ_{123} and δ_{321}



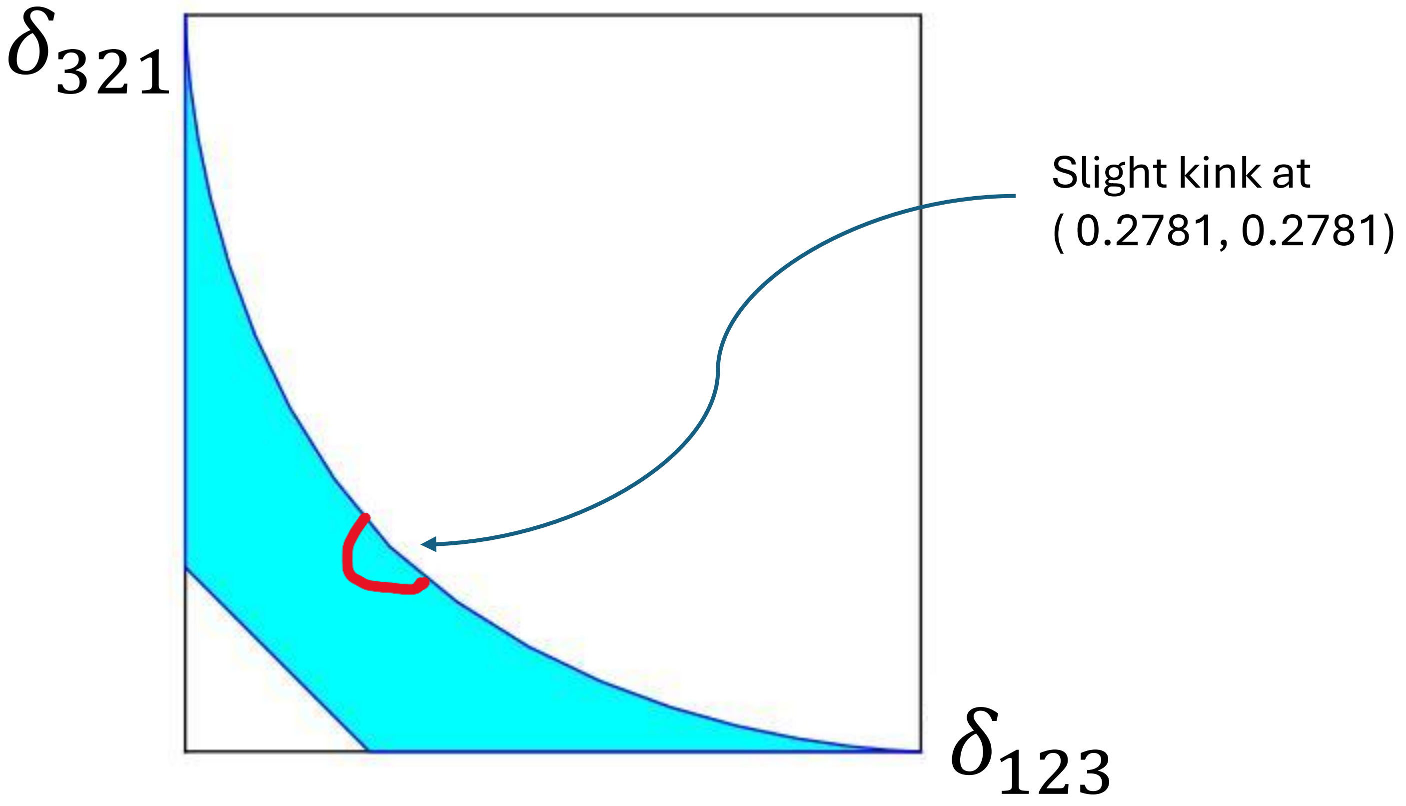
Joint Packing Density for δ_{123} and δ_{321}



Joint Packing Density for δ_{123} and δ_{321}



Joint Packing Density for δ_{123} and δ_{321}



This path in R^2 is not the image of any continuous path among permutons.

Joint Packing Density for δ_{123} and δ_{321}

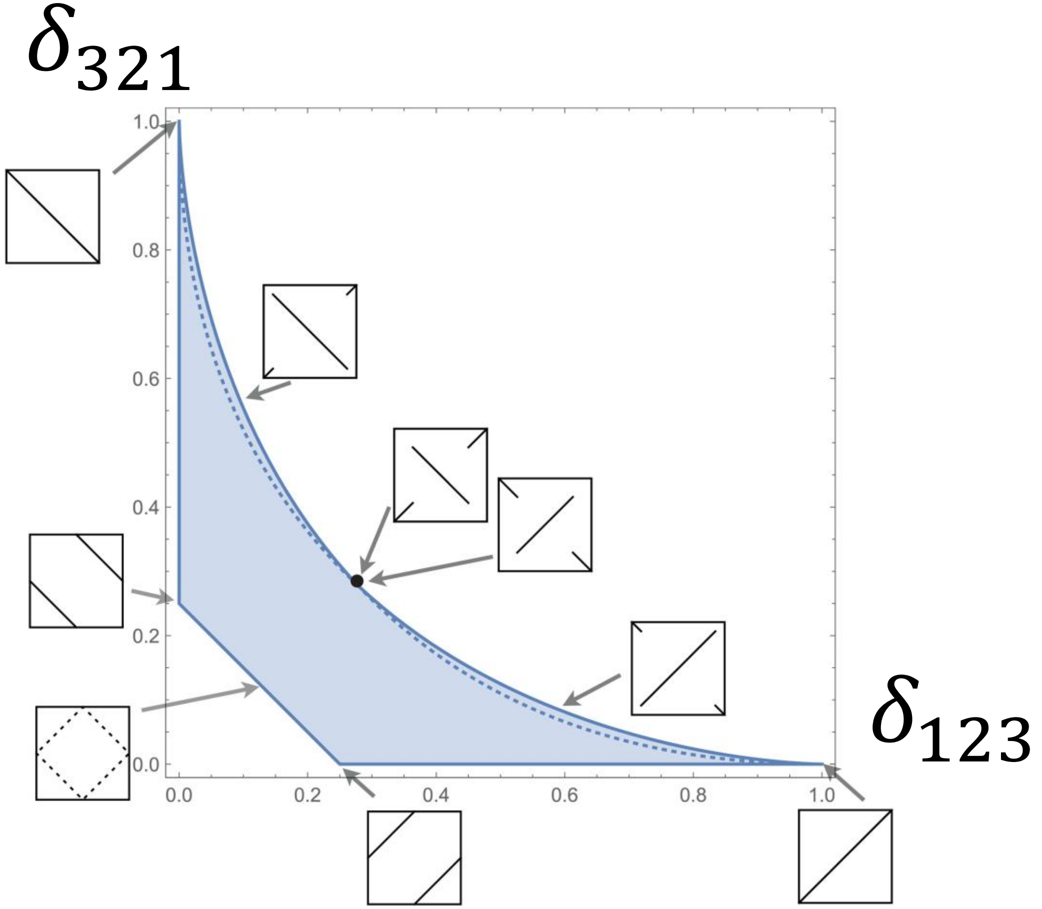
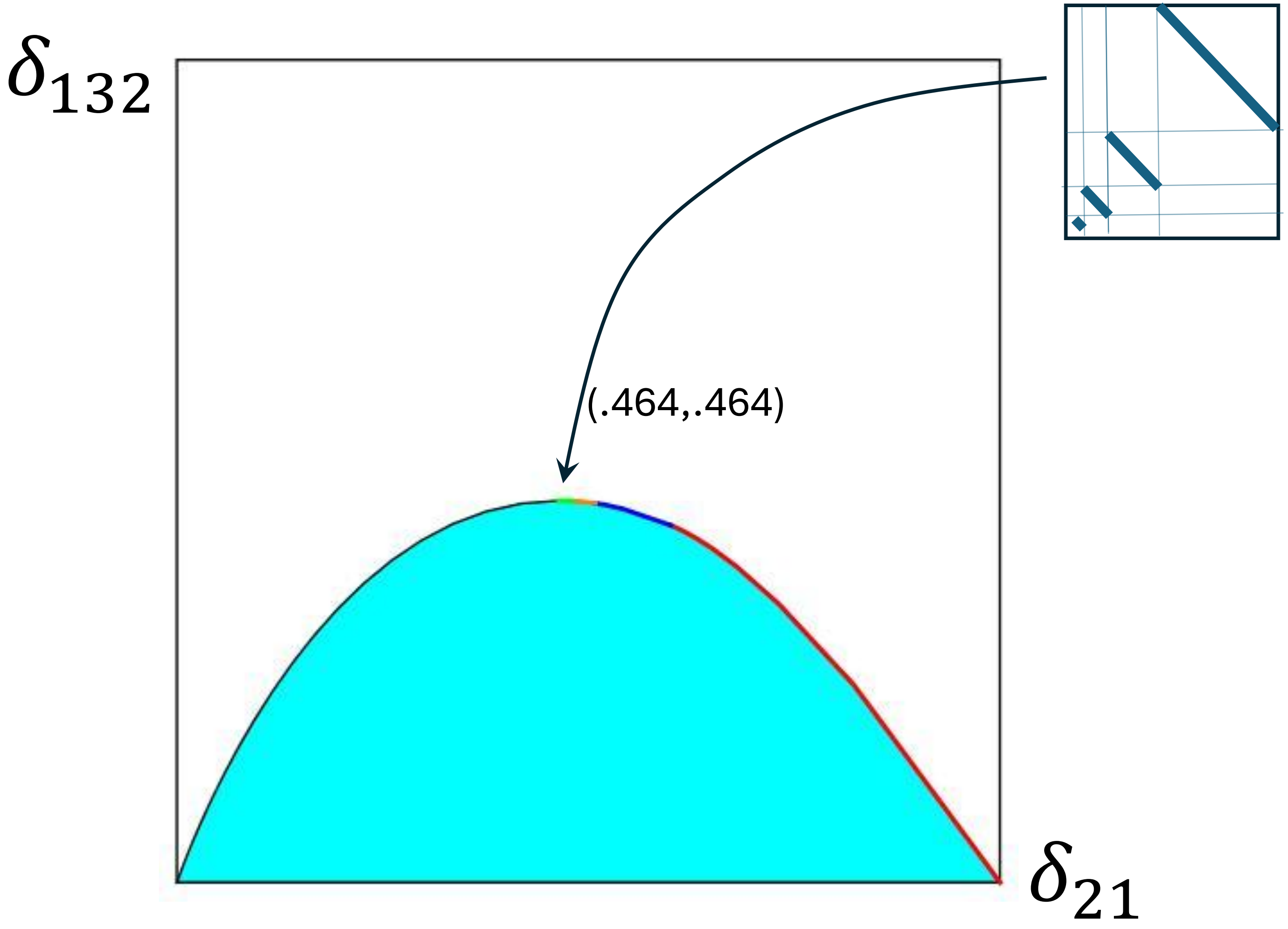
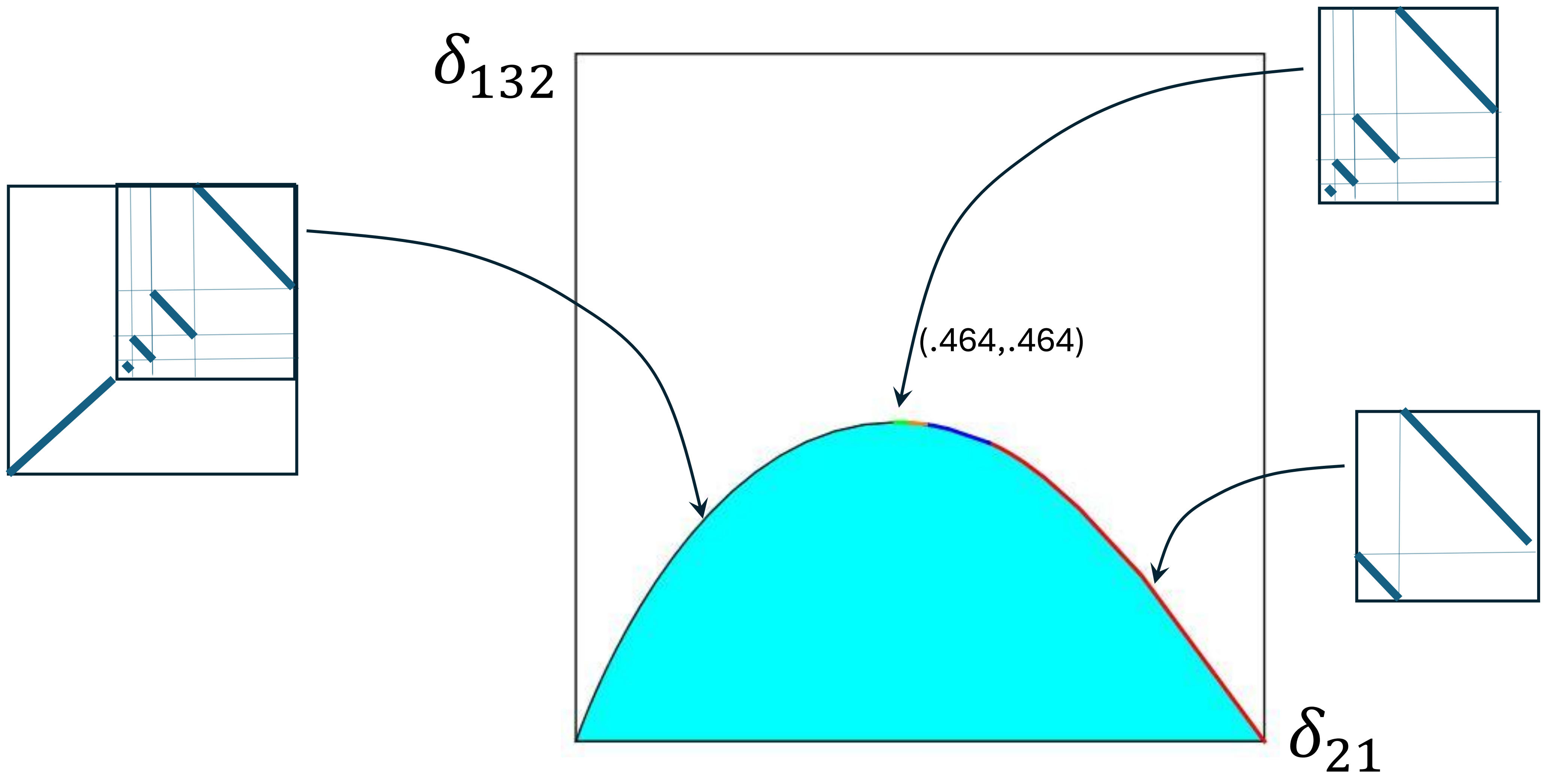


Image from Figure 9 of
Permutations with fixed pattern densities
Richard Kenyon, Daniel Král', Charles Radin, Peter Winkler
Random Structures & Algorithms 56(1) Jan. 2020

Joint Packing Density for δ_{21} and δ_{132}



Joint Packing Density for δ_{21} and δ_{132}



Joint Packing Density for (δ_{12}) and $(\delta_{123} + \delta_{213})$

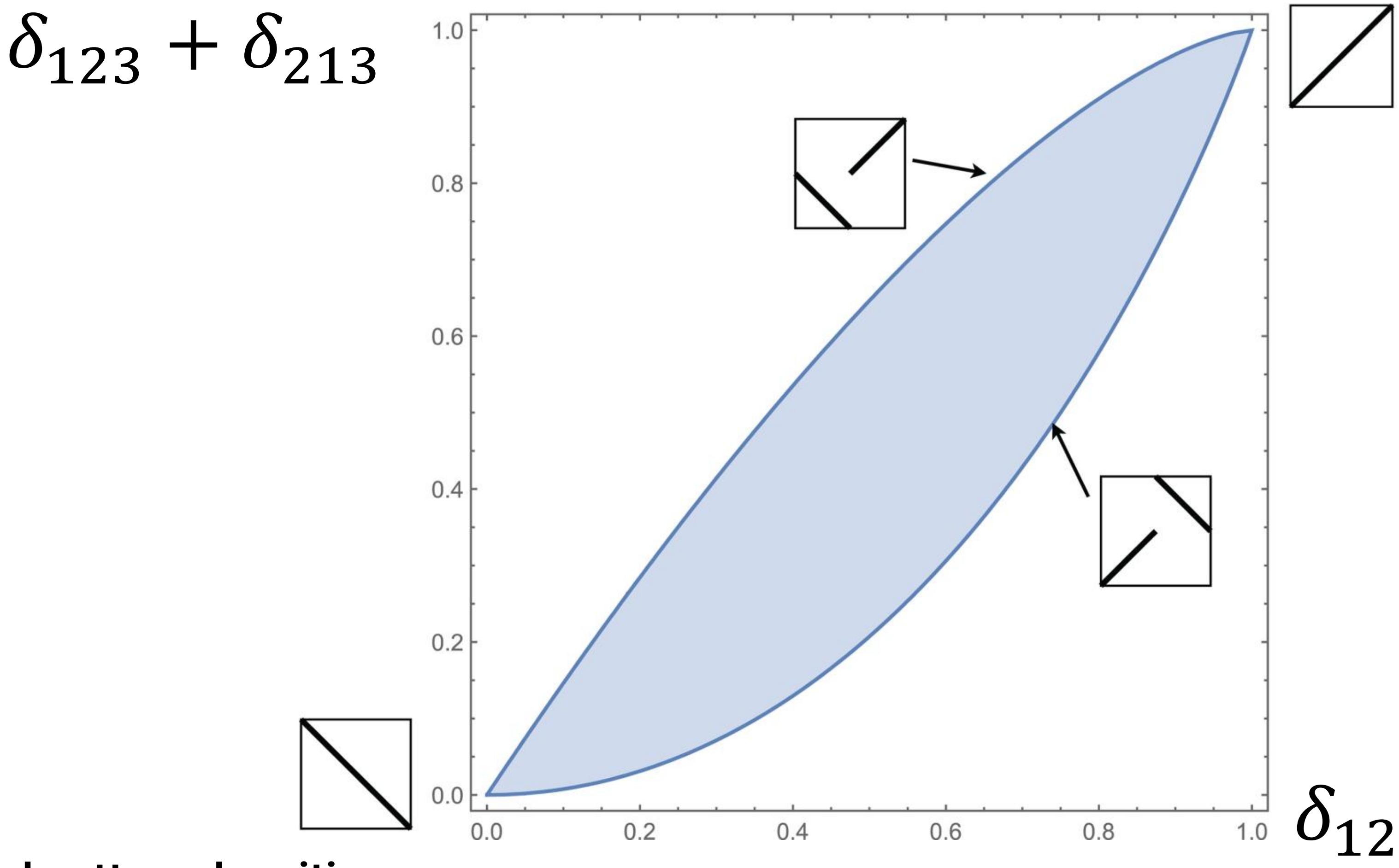
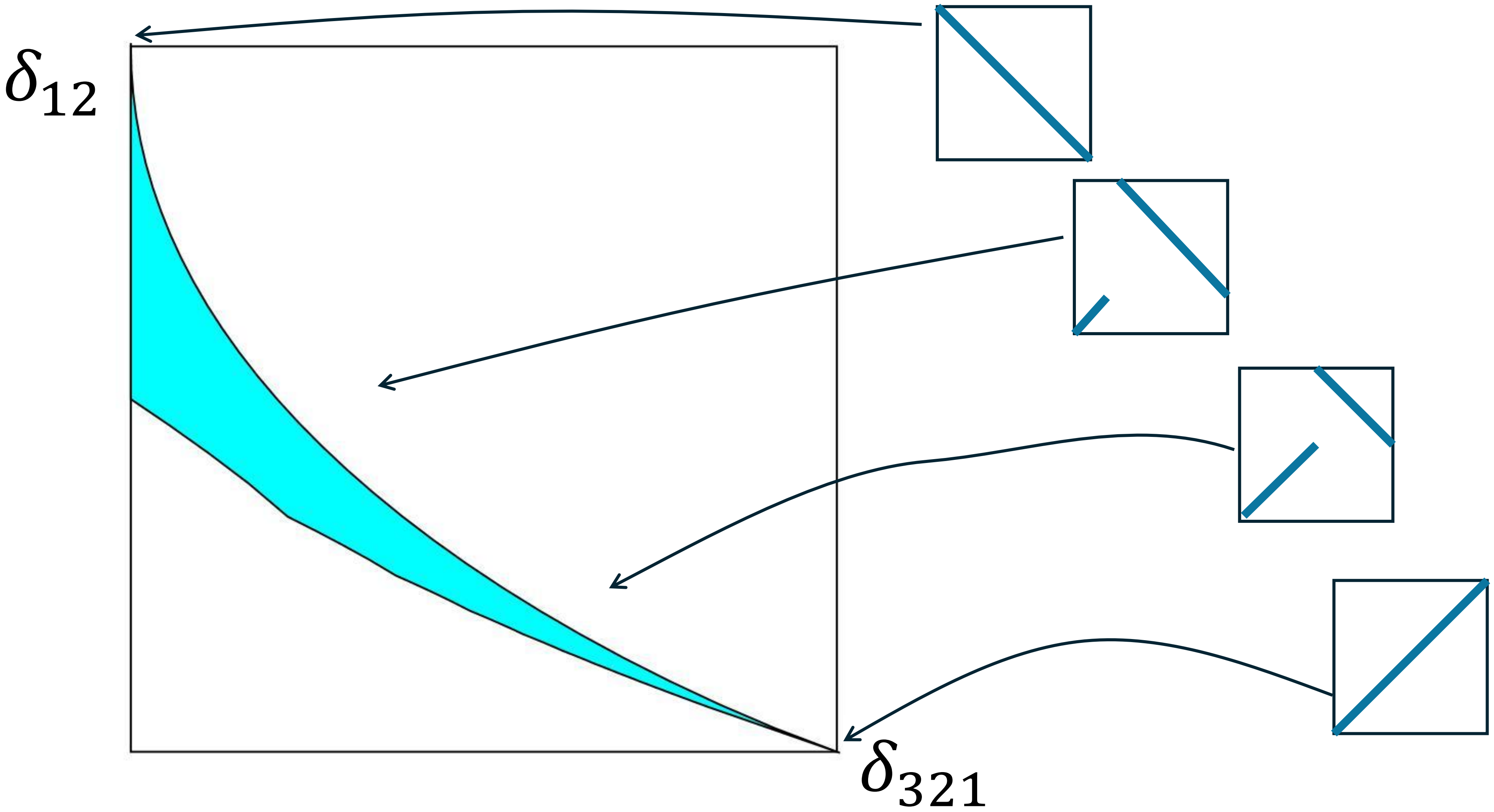
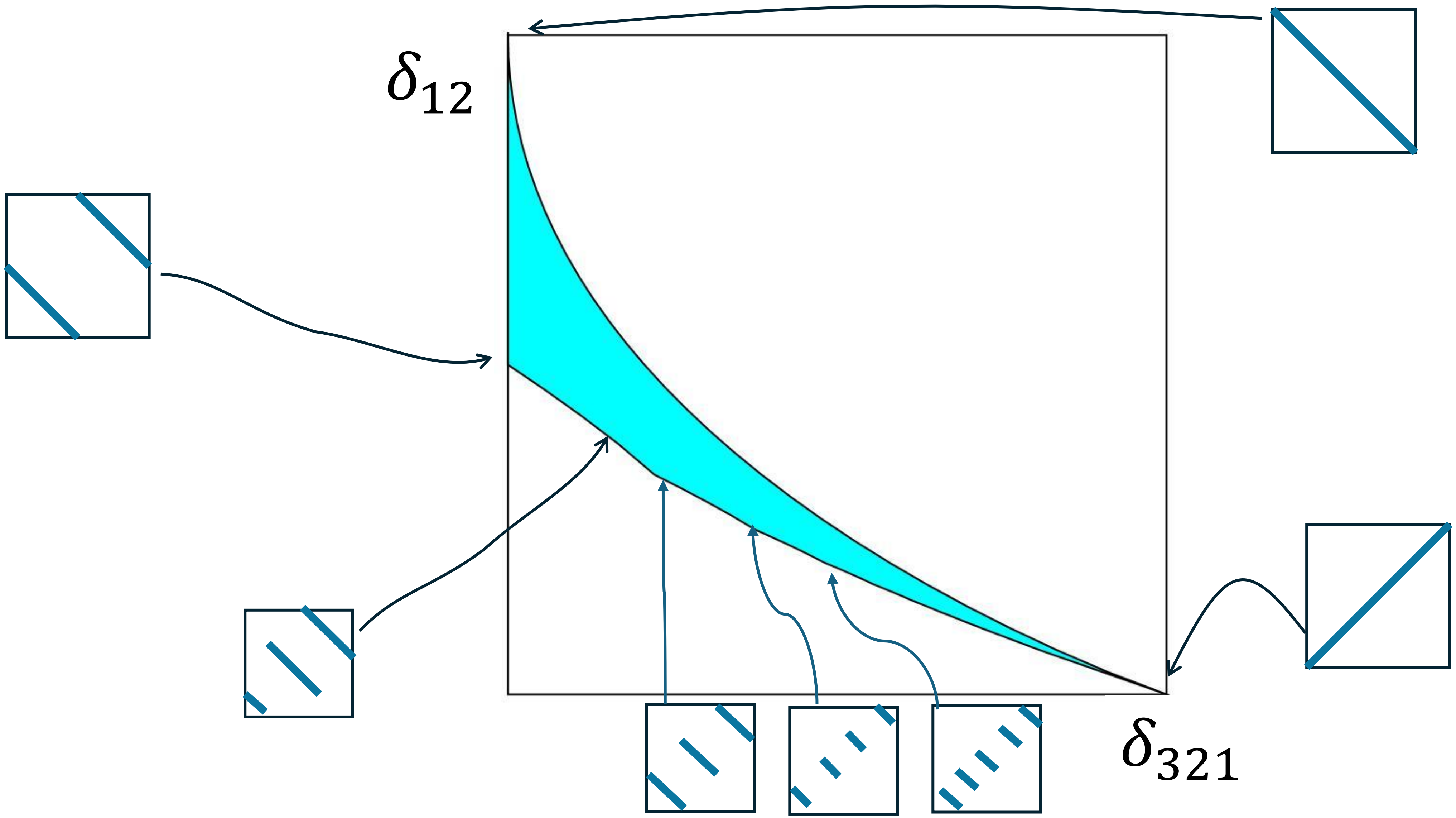


Image from Figure 4 of
Permutations with fixed pattern densities
Richard Kenyon, Daniel Král', Charles Radin, Peter Winkler
Random Structures & Algorithms 56(1) Jan. 2020

Joint Packing Density for δ_{12} and δ_{321}



Joint Packing Density for δ_{12} and δ_{321}



Joint Packing Density for δ_{12} and δ_{321}

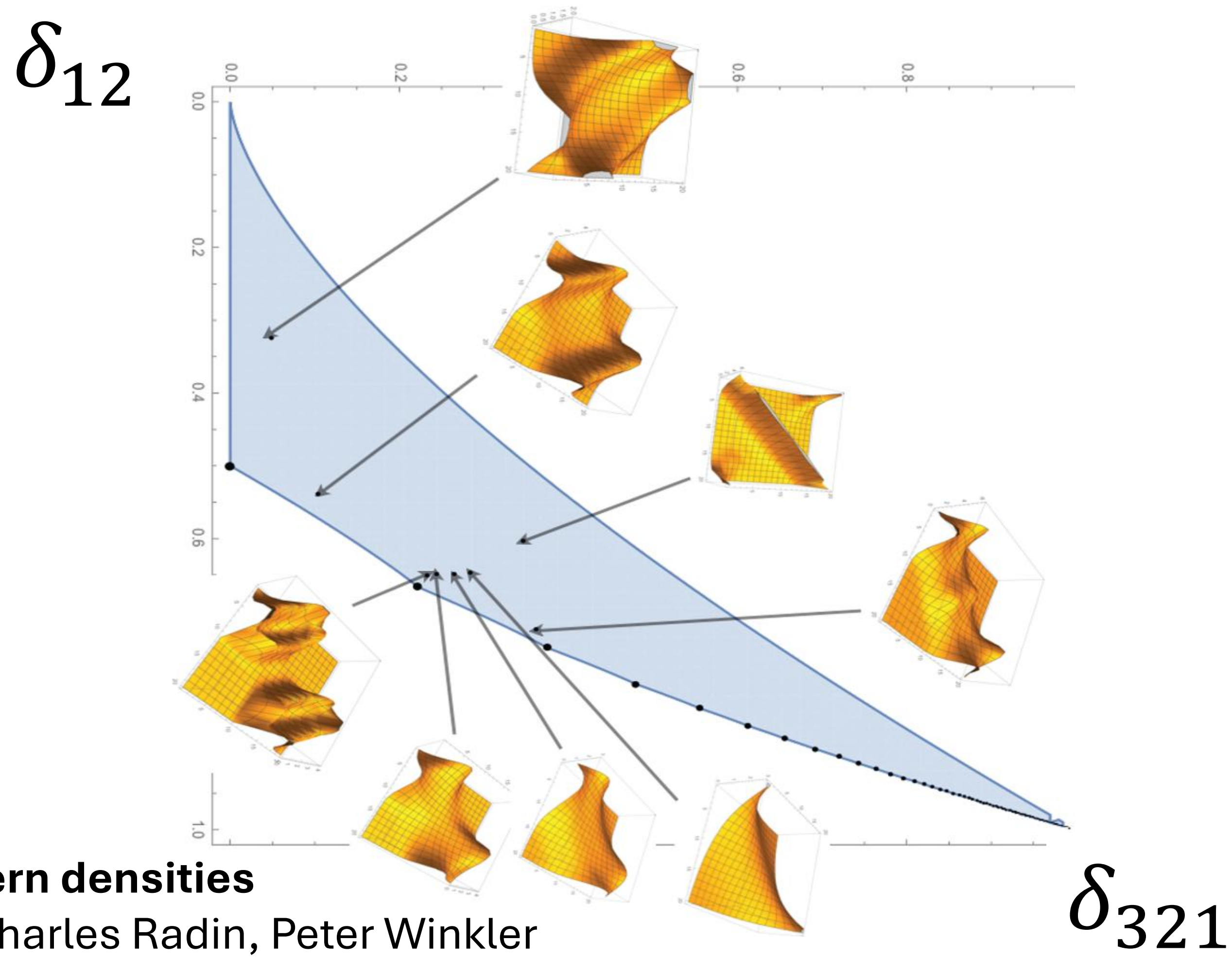


Image from Figure 7 (modified)
Permutations with fixed pattern densities
Richard Kenyon, Daniel Král', Charles Radin, Peter Winkler
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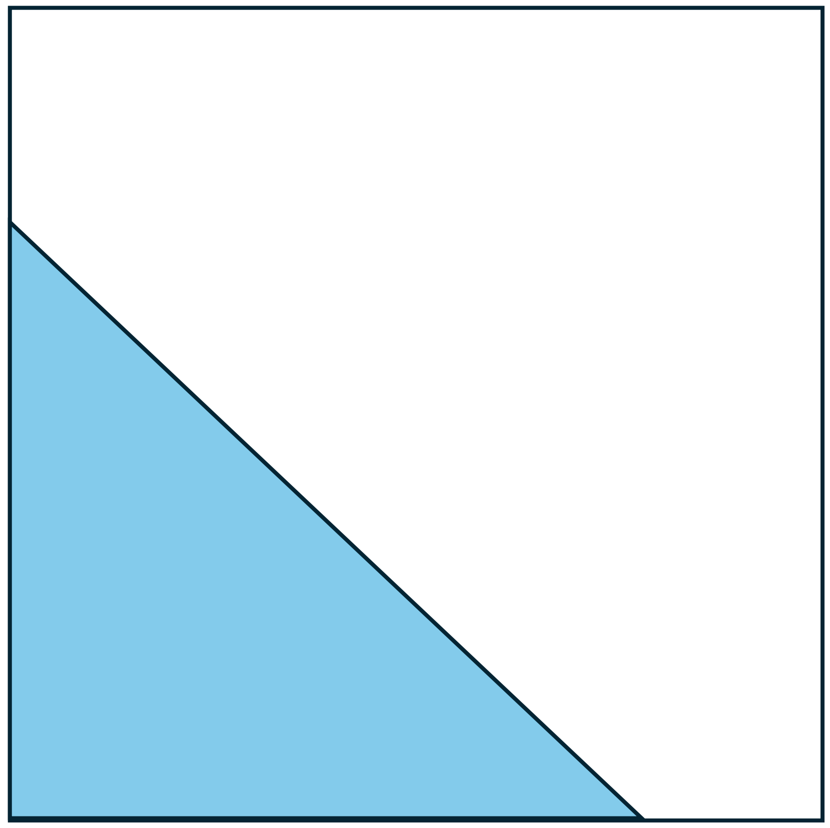
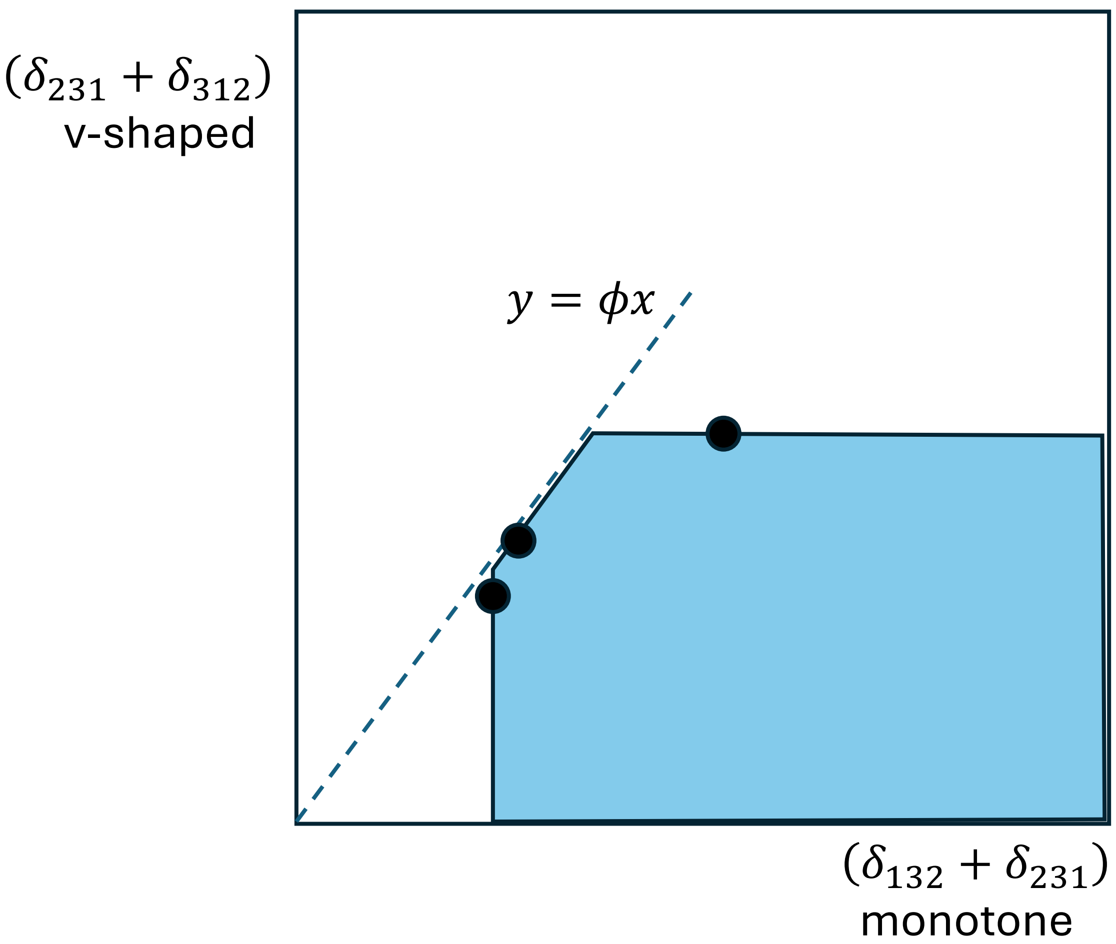
Joint Packing Density for $(\delta_{132} + \delta_{213})$ and $(\delta_{231} + \delta_{312})$



Joint Packing Density for $(\delta_{132} + \delta_{231})$ and $(\delta_{231} + \delta_{312})$



The Layered Version

Today we will deal with the special case of LAYERED PERMUTATIONS.

A LAYERED PERMUTATION is one that contains no 231 or 312 patterns.

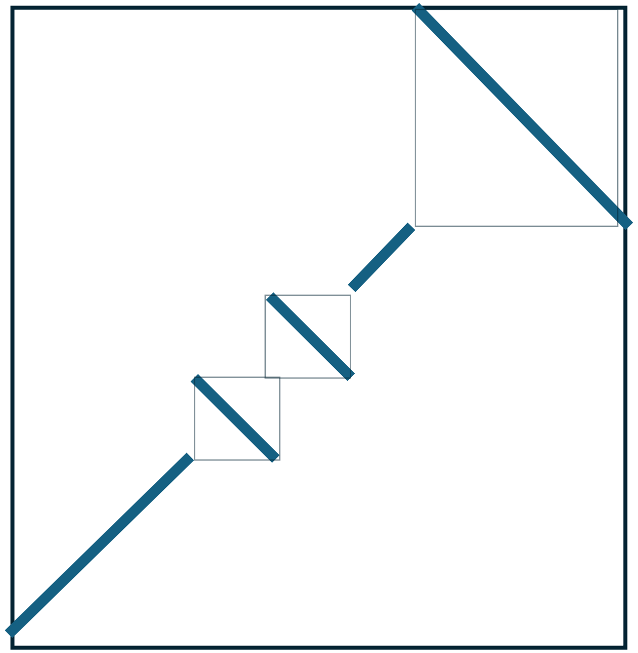
In dealing with layered permutations, we will omit those components of the packing vector, which becomes

$$v = (\delta_{123}, \ \delta_{132}, \ \delta_{213}, \ \delta_{321}) \in R^4.$$

Which of these vectors can be limits of packing vectors of layered permutations?

The answer is a compact subset of R^4, contained in the 3-d standard simplex.

Layered Permutons



$\langle x_1, x_2, x_3 \rangle$ = sizes of DOWN boxes, in decreasing order. In this case, $\left\langle \frac{1}{3}, \frac{1}{9}, \frac{1}{9} \right\rangle$.

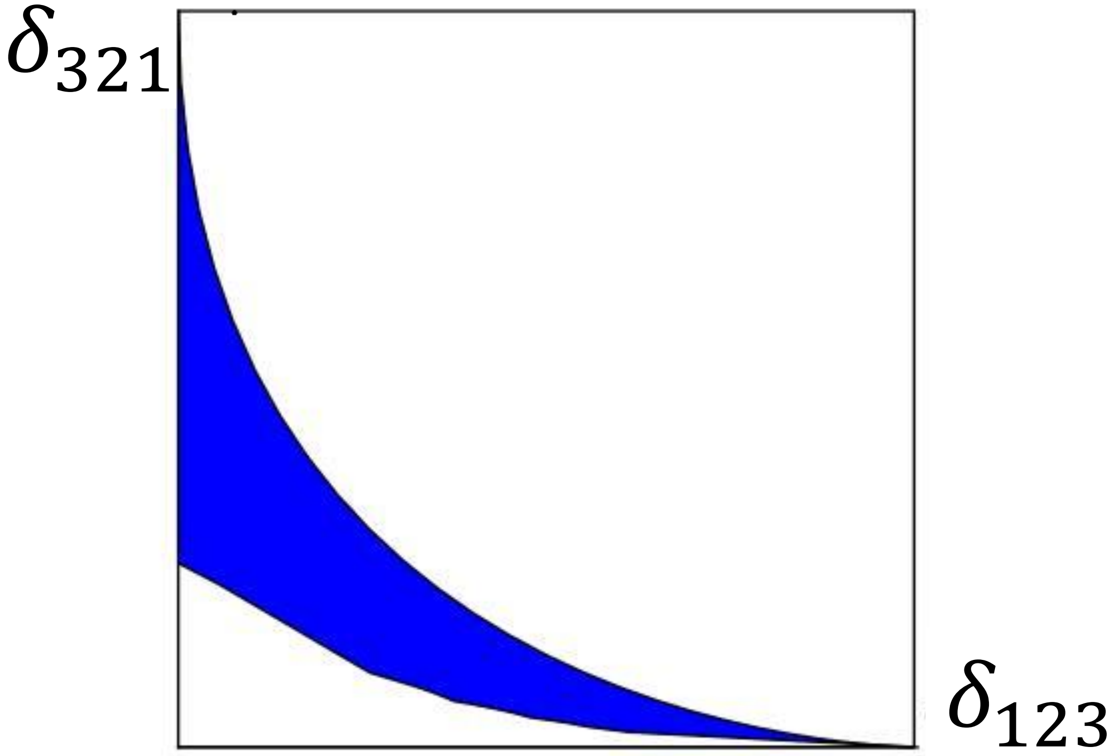
For δ_{123} and δ_{321}, only the *x*'s matter---NOT their order, or anything about the UP boxes.

$$\delta_{123} = 1 - 3\sum x_i^2 + 2\sum x_i^3$$

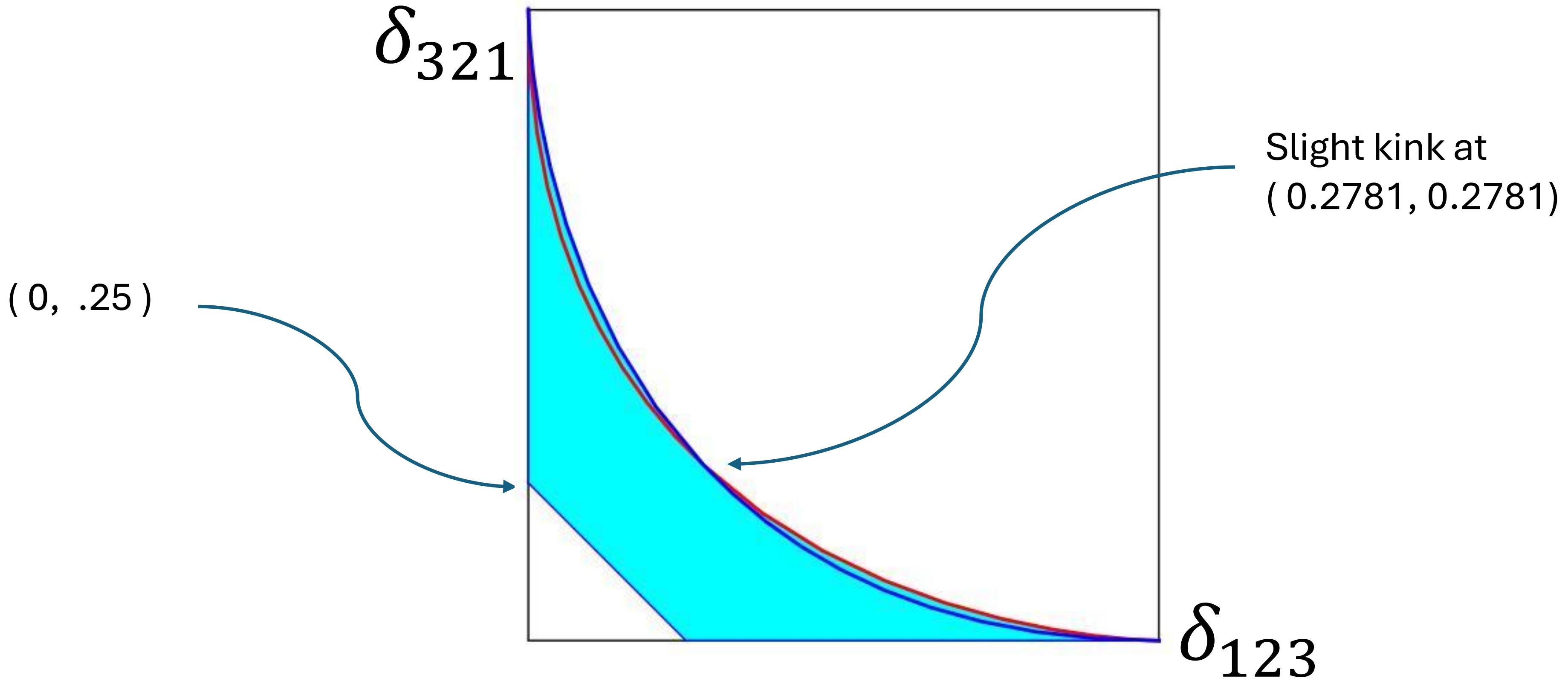
$$\delta_{321} = \sum x_i^3$$

For δ_{132} and δ_{213}, order and placement matter.

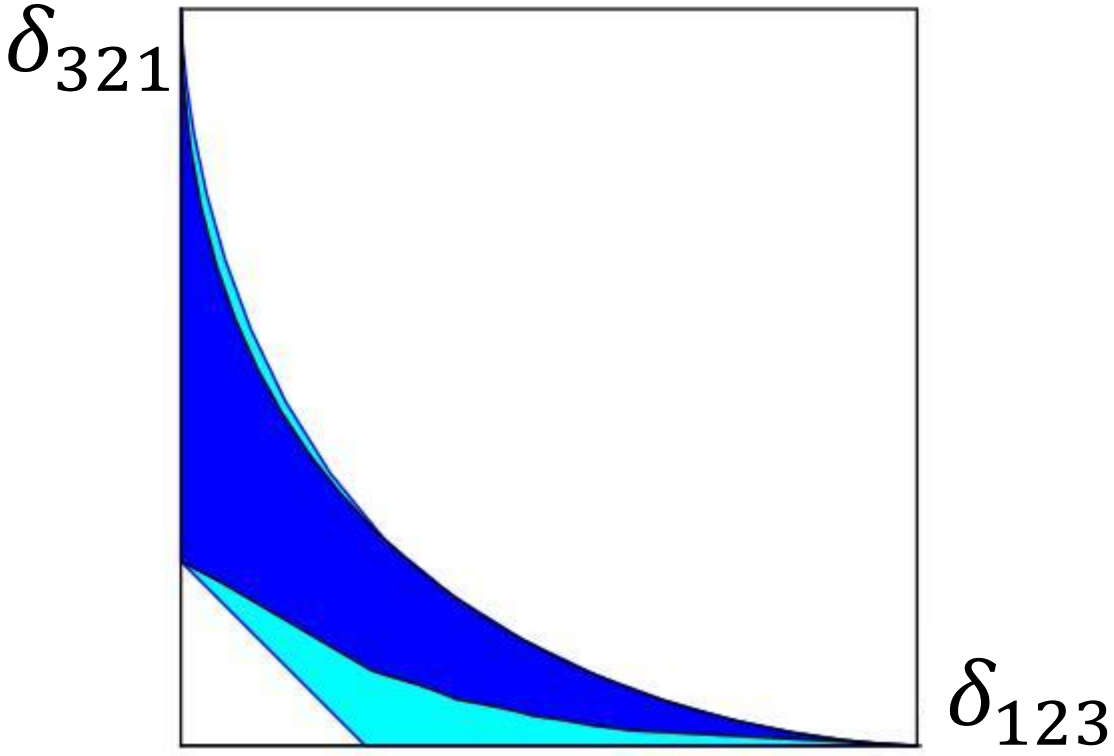
Layered Version: The Main Diagram



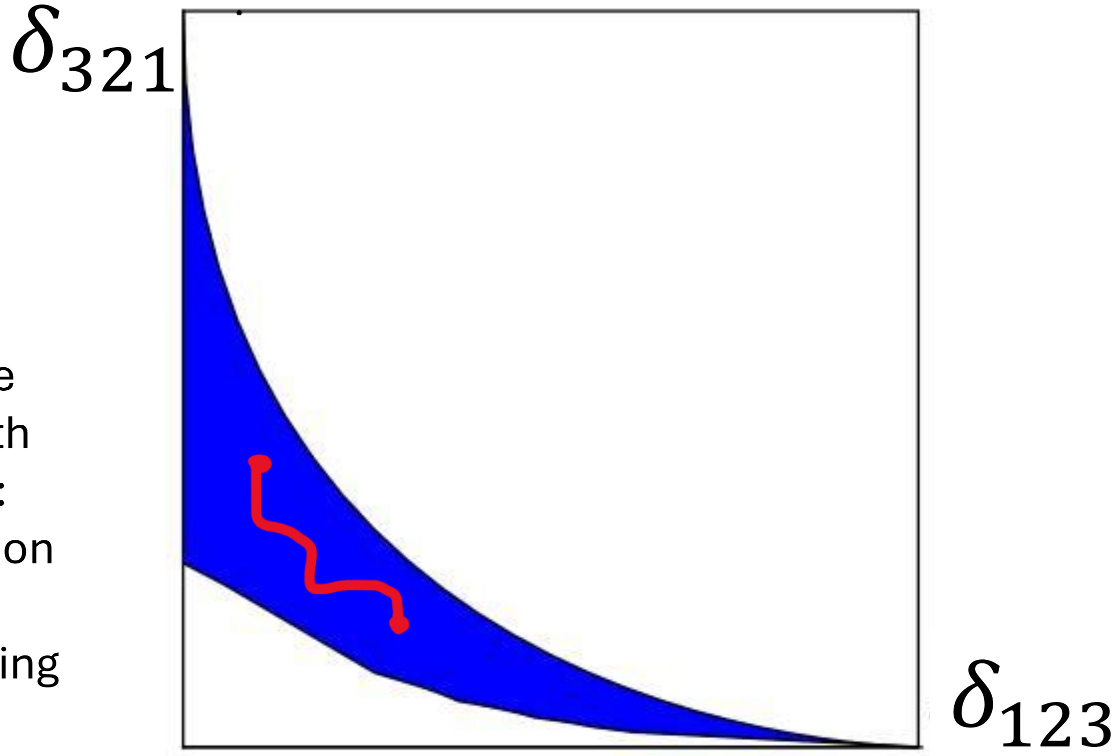
Joint Packing Density for δ_{123} and δ_{321}



Layered Version: The Main Diagram

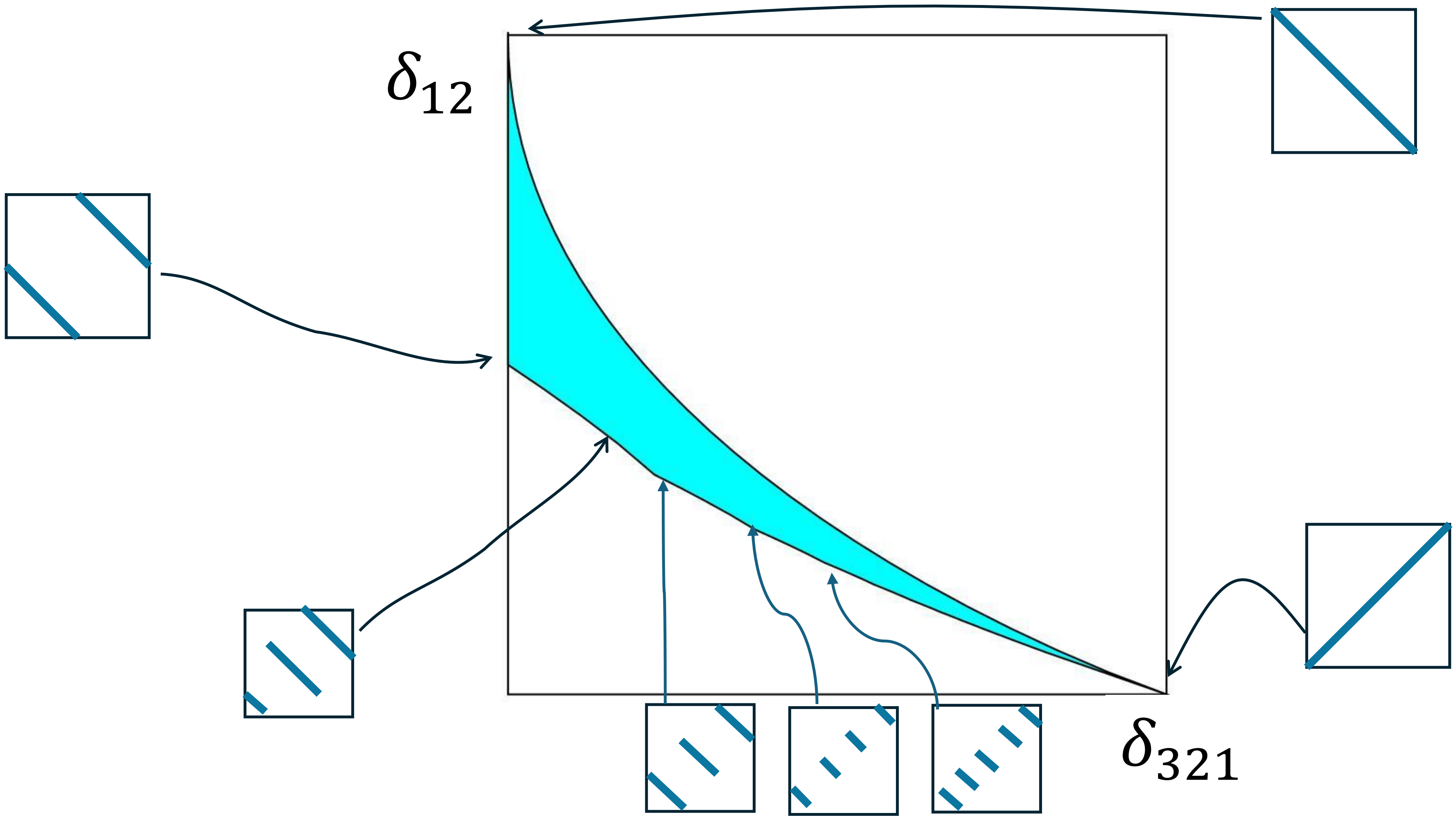


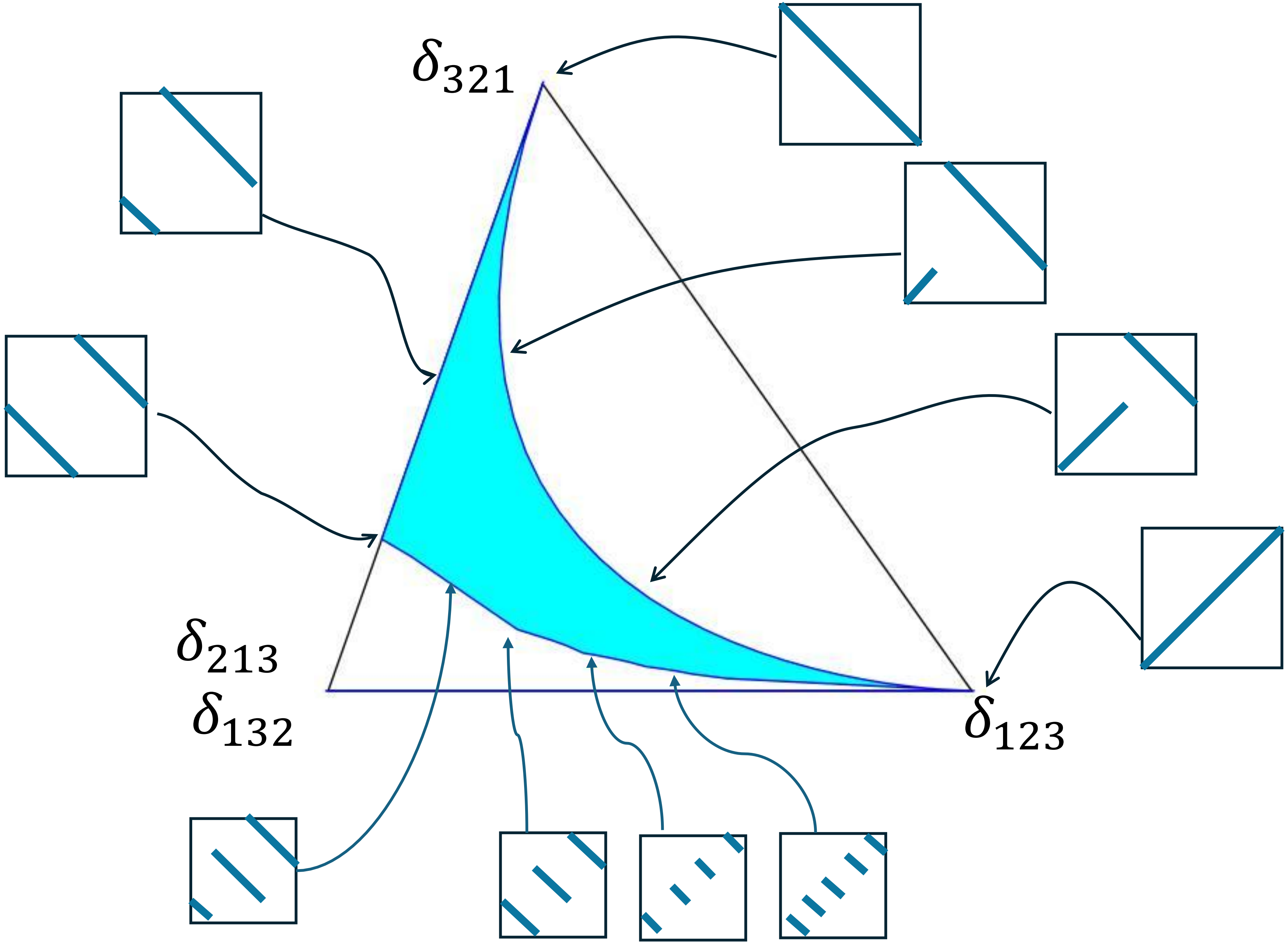
Layered Version: The Main Diagram

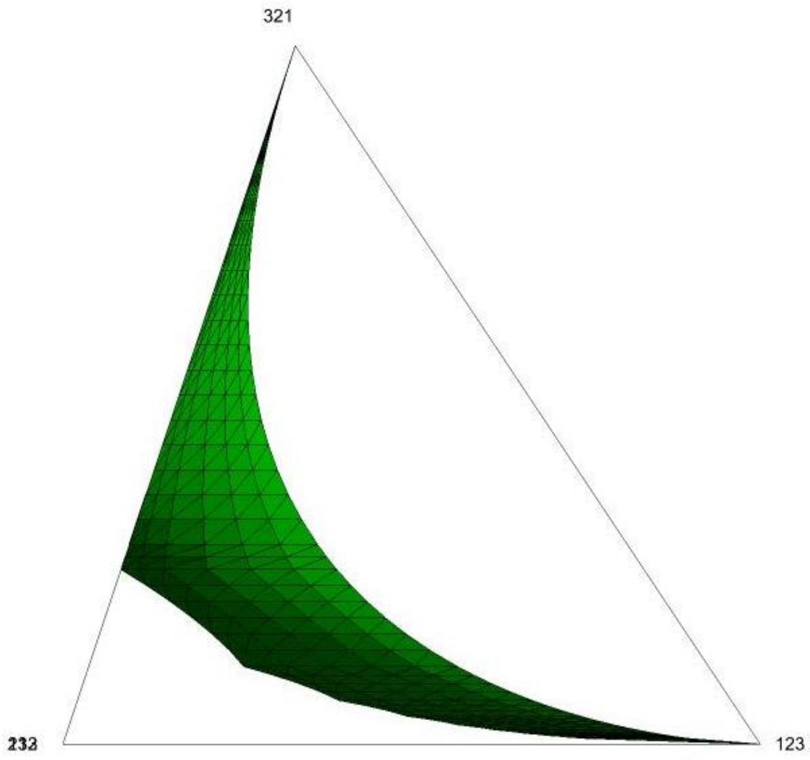


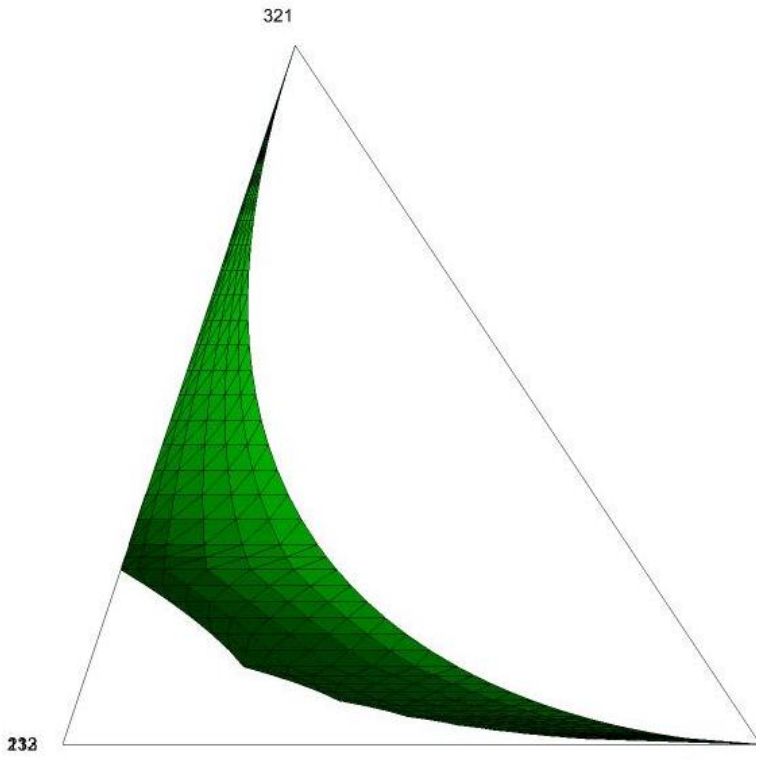
Every path in this set is the image of a continuous path among permutons. Proof: Each point in the blue region corresponds to a unique permuton whose decreasing layers have sizes
w, wz, ..., wz, (1-nw)z
for w, z in [0,1].

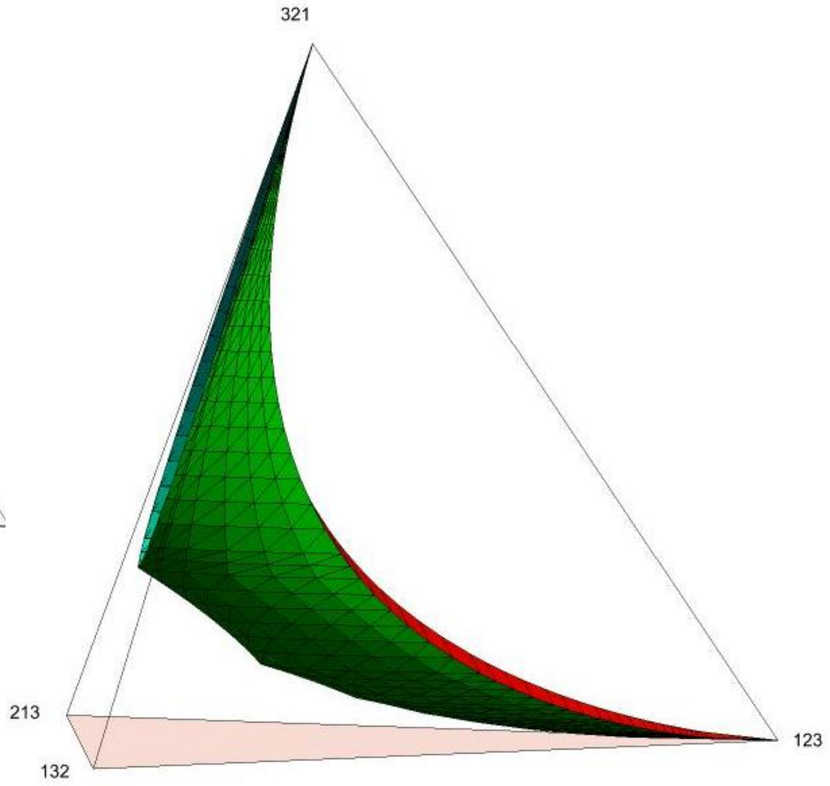
Joint Packing Density for δ_{12} and δ_{321}

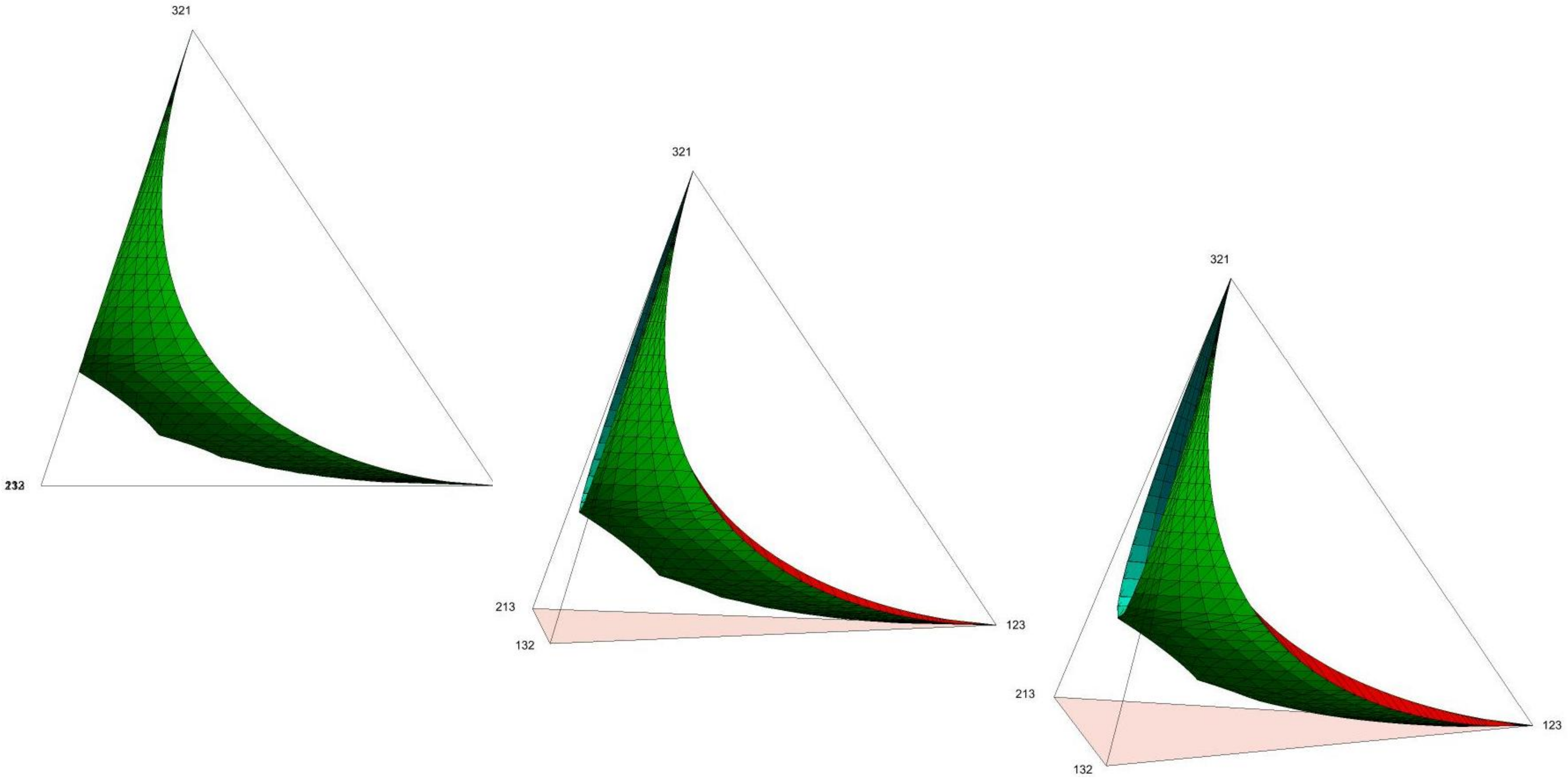



δ_{321}
δ_{213}
δ_{132}
δ_{123}


321
232
123

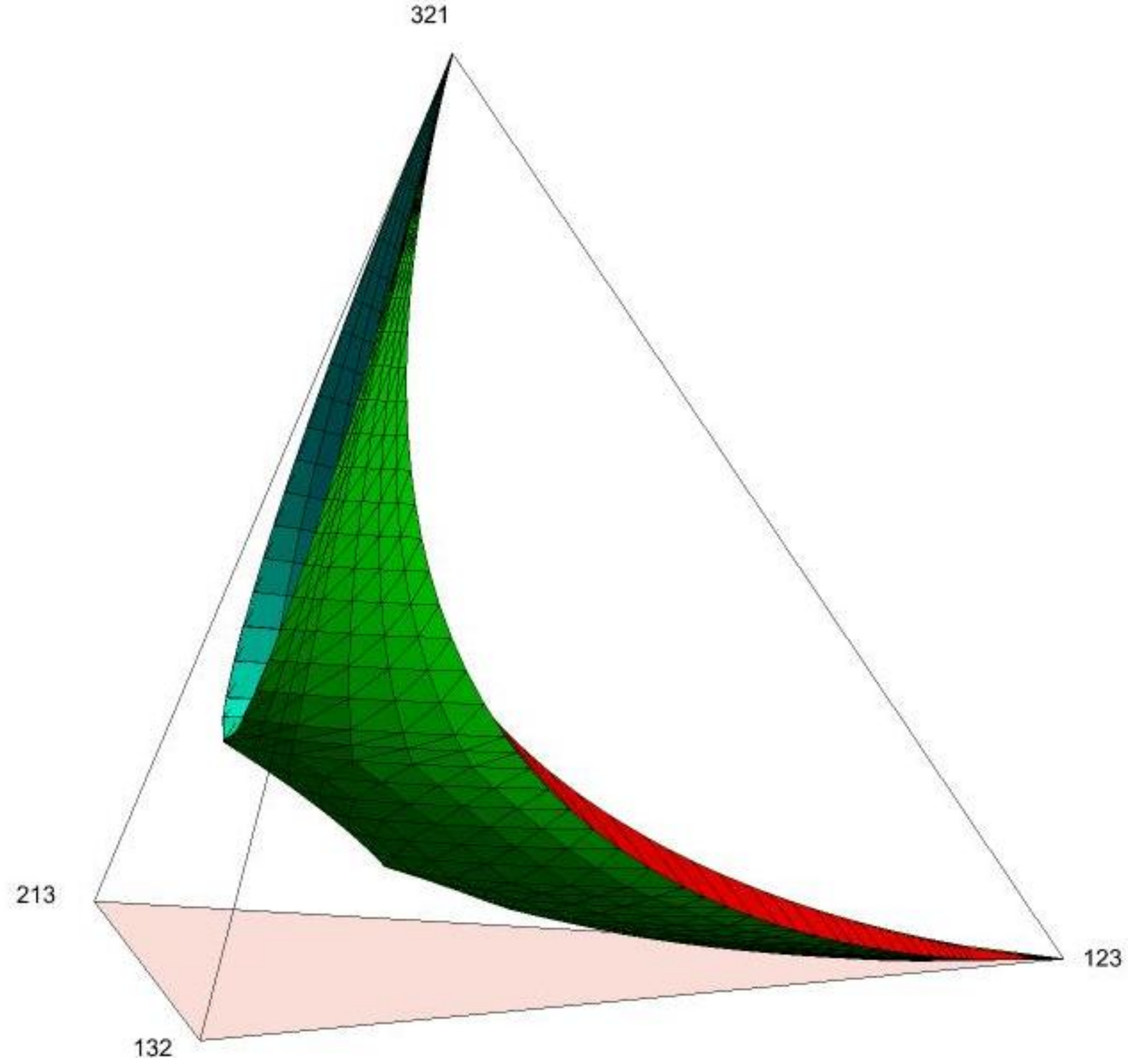

321


321
213
132
123

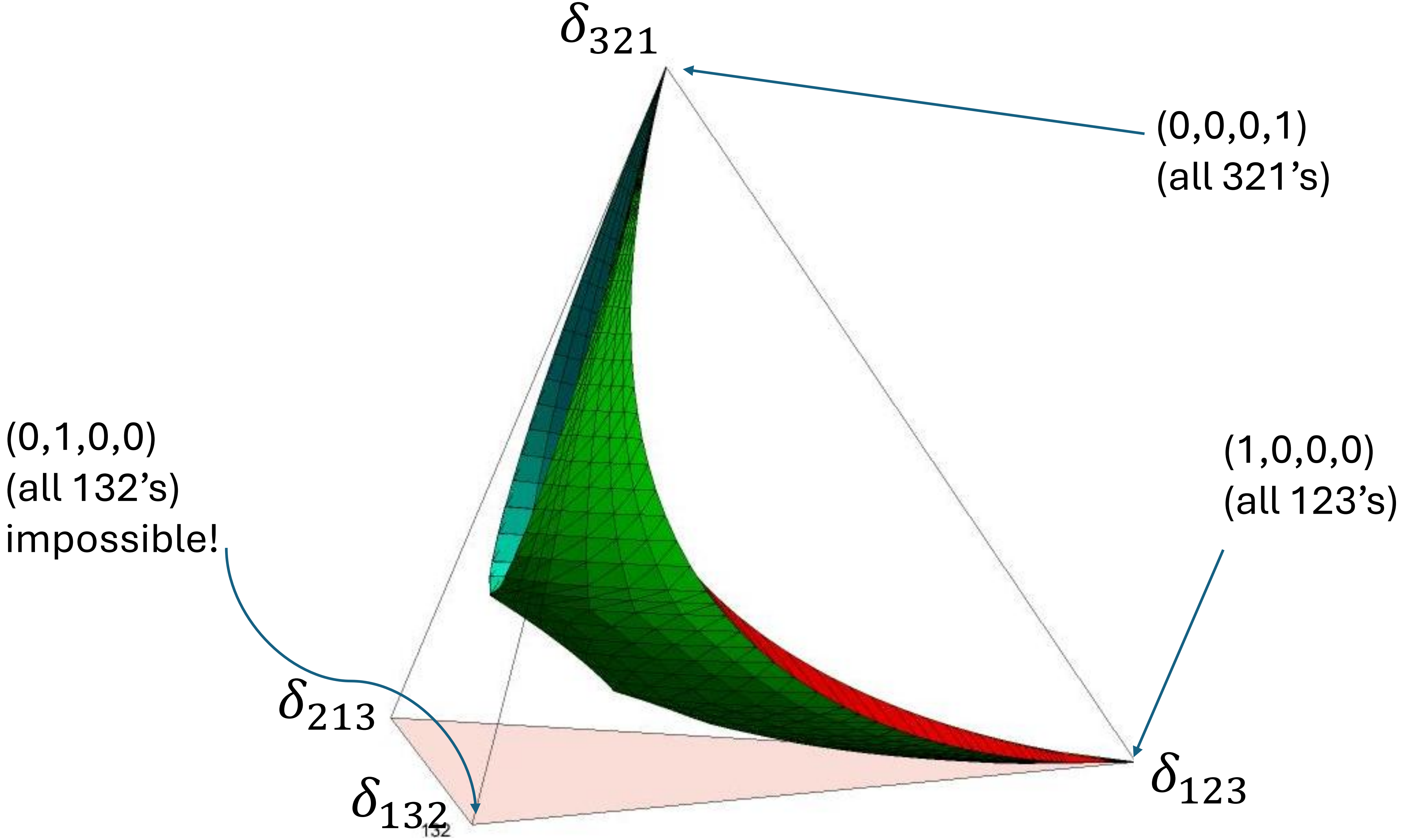


321
321
213
132
123
321
213
132
123

Great Limit Shape (layered version)



Great Limit Set (layered version)



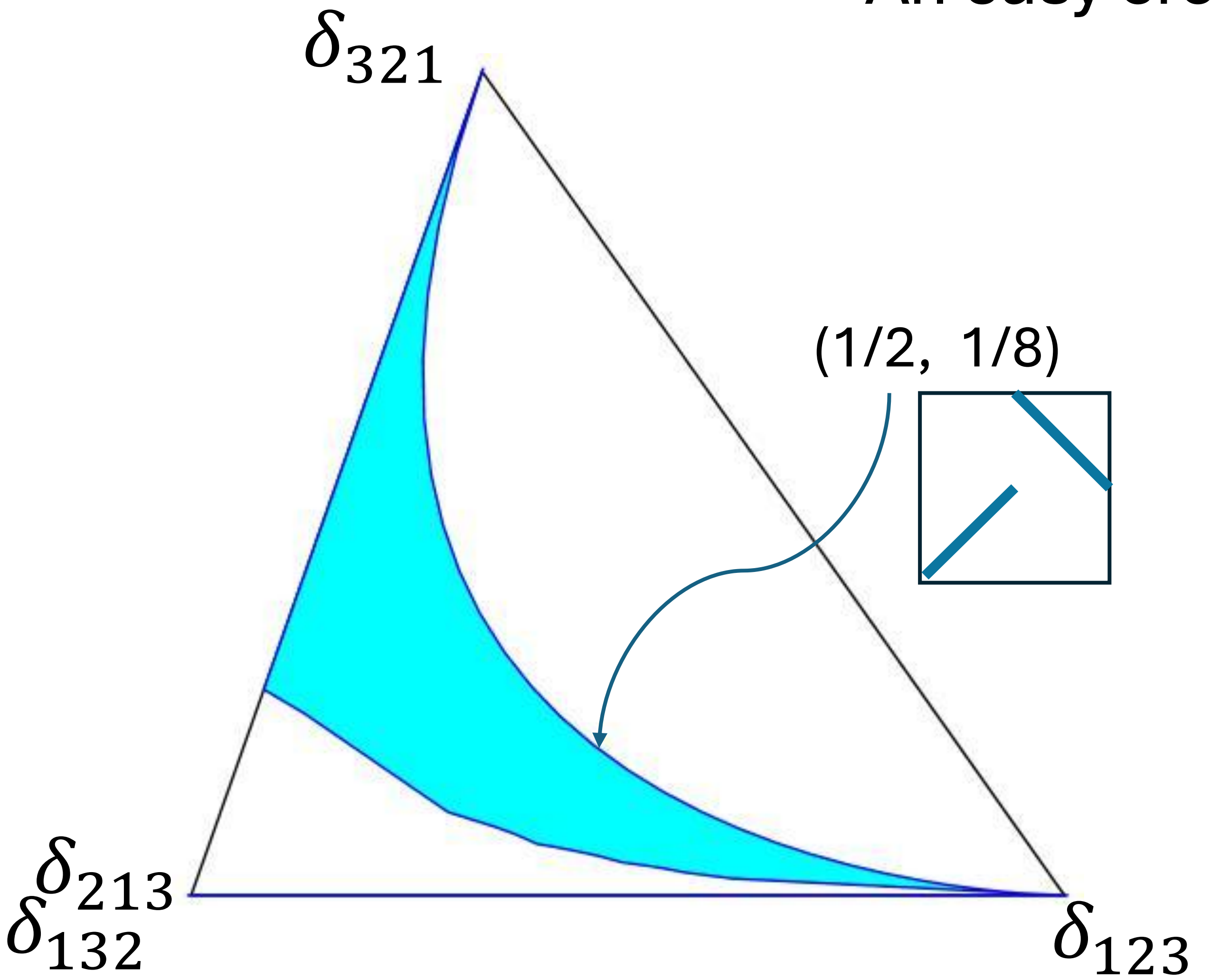
The Cross Sections

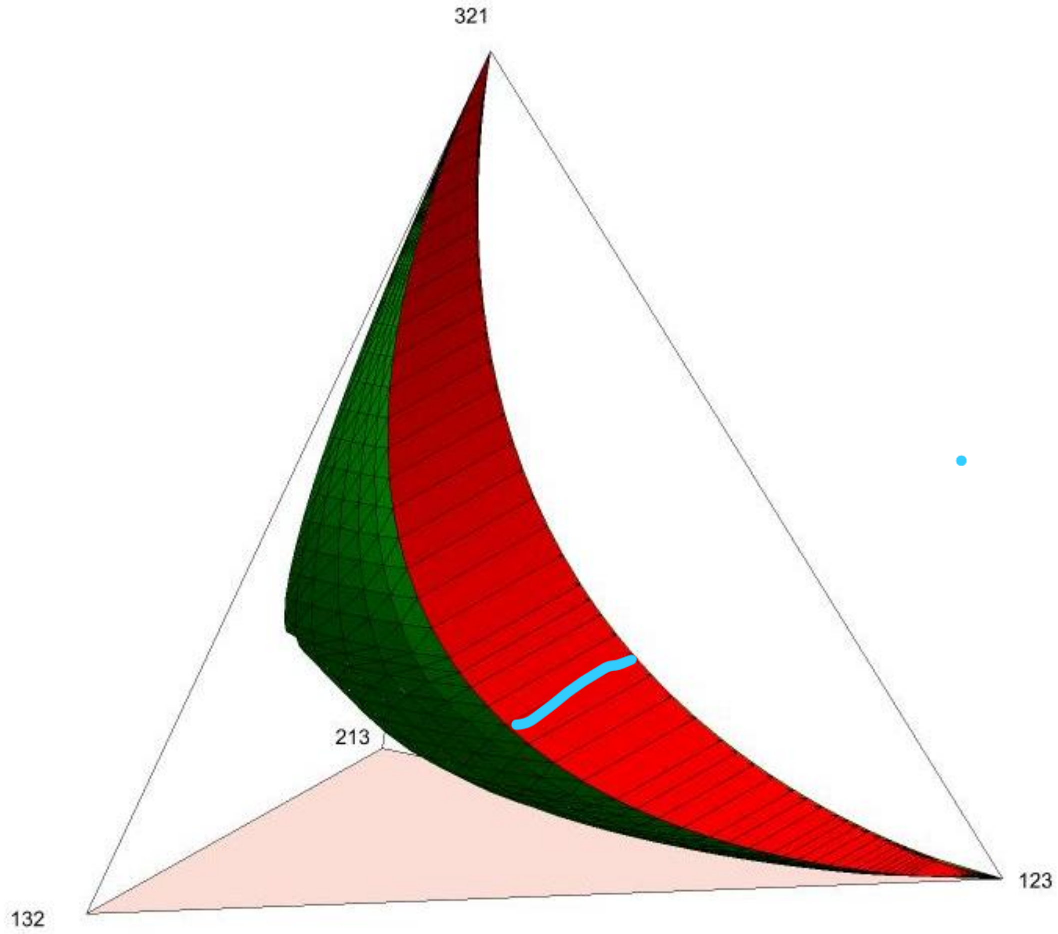
If we knew --- for every point in the main diagram --- what are the possible values of $(\delta_{132}, \delta_{213})$, then we would know the entire shape.

Because $\delta_{213} = 1 - \delta_{123} - \delta_{321} - \delta_{132}$, and both δ_{123} and δ_{321} are given by the point we have chosen, we just need to know the possible values of δ_{132}.

Start with points on the boundary of the main diagram.

An easy cross section

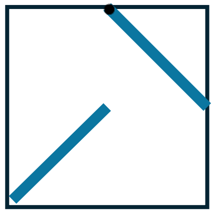




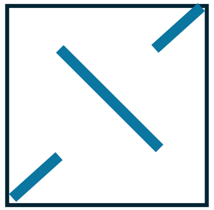
The cross section at (1/2,1/8)

Suppose $\delta_{123} = 1/2$ and $\delta_{321} = 1/8.$ Those values define a point on the upper boundary of the main diagram. They leave 3/8 for δ_{132} and δ_{213} .

Fact: δ_{132} can take any value in [0, 3/8].



has packing vector (1/2, 3/8, 0, 1/8).

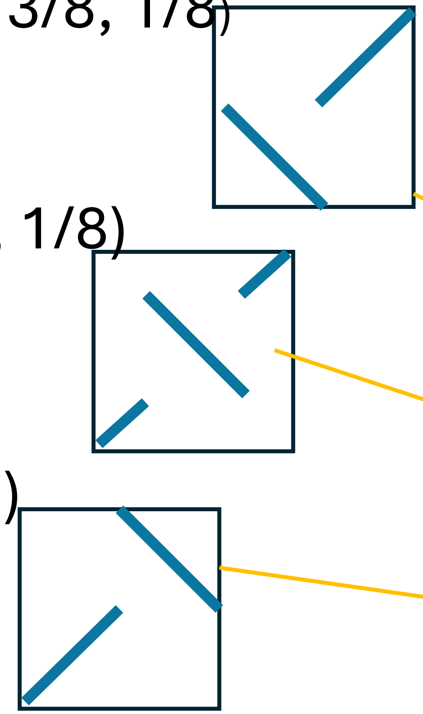


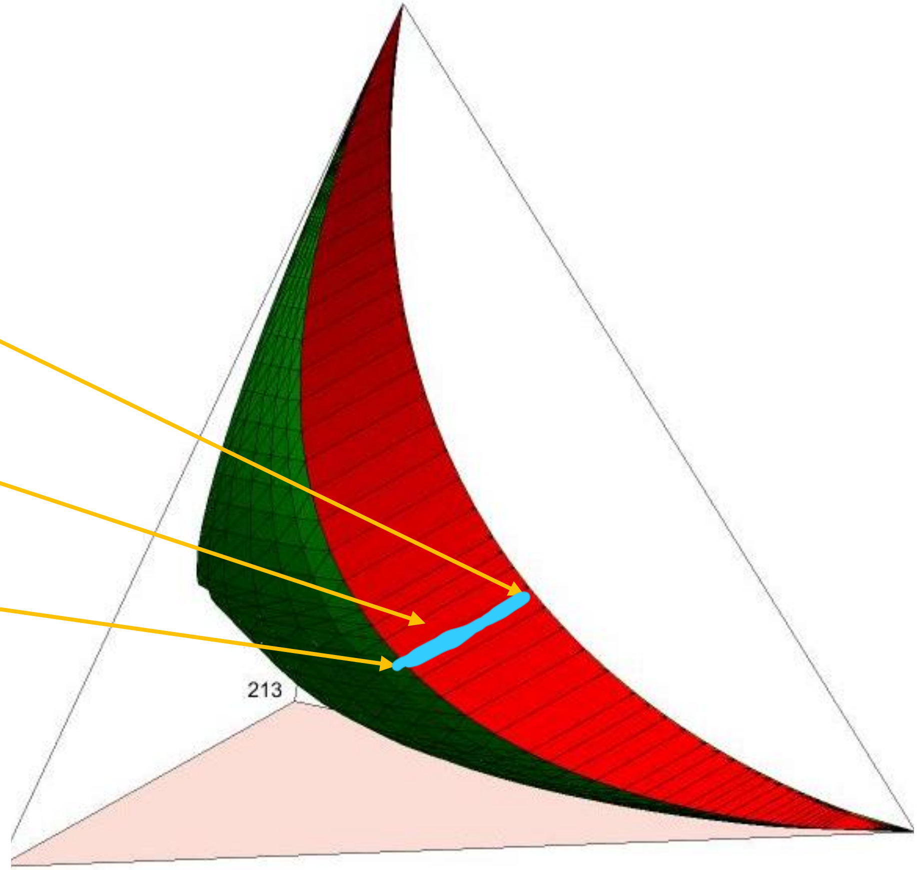
has packing vector (1/2, 3/16, 3/16, 1/8).

has packing vector (1/2, 0, 3/8, 1/8).

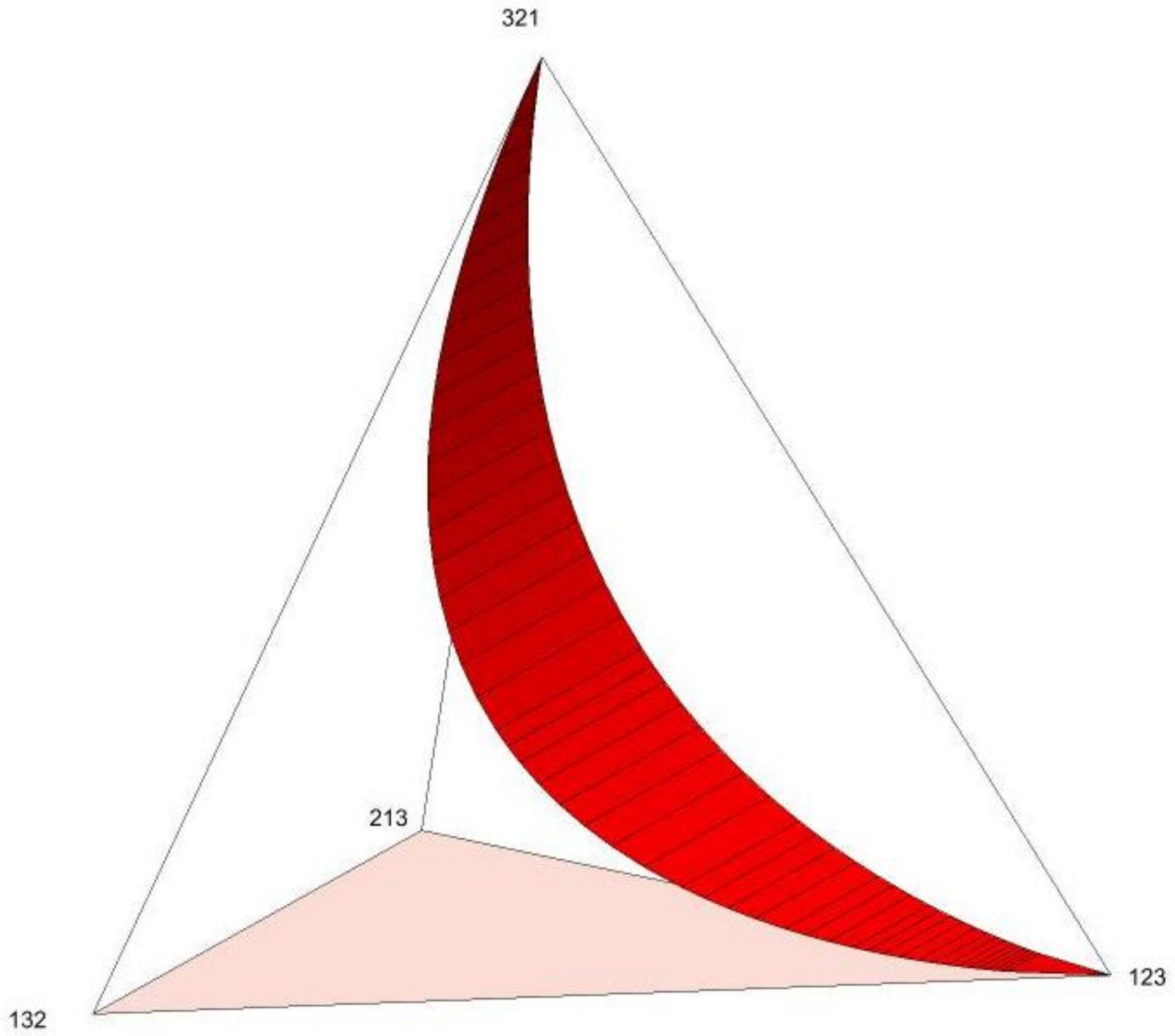
The “up” bits give us slack to adjust δ_{132} to any value we like.

An easy cross section

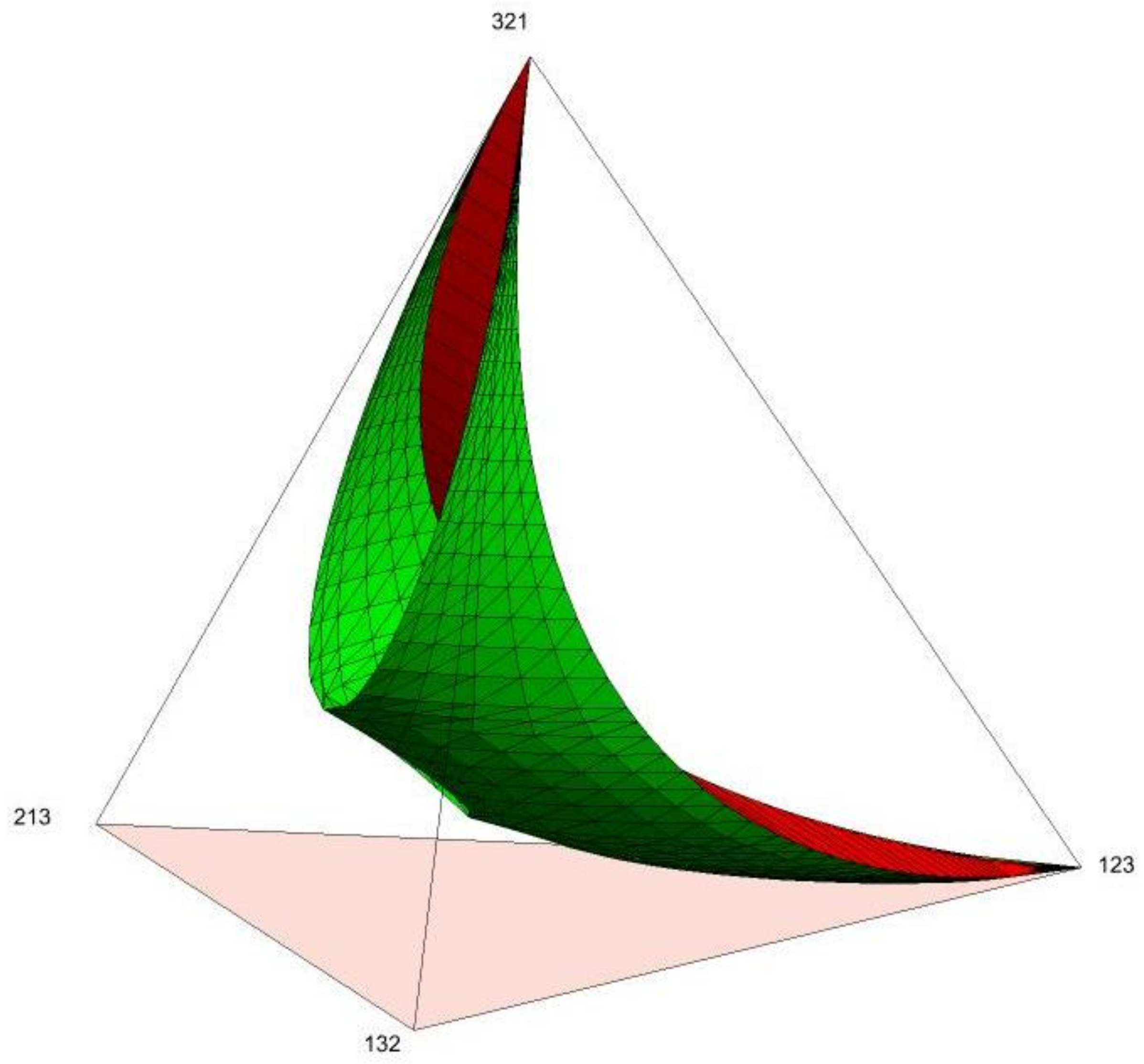


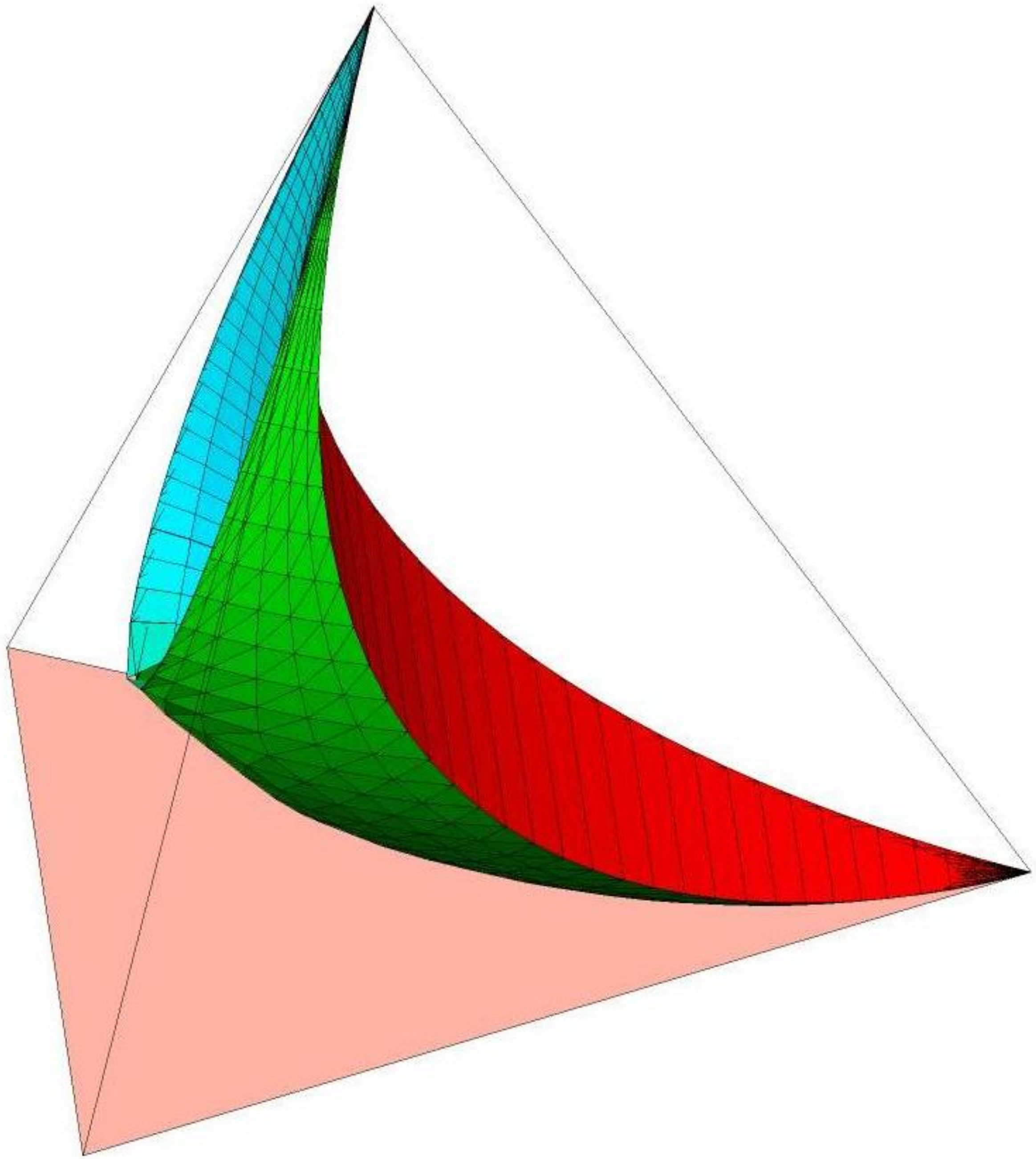


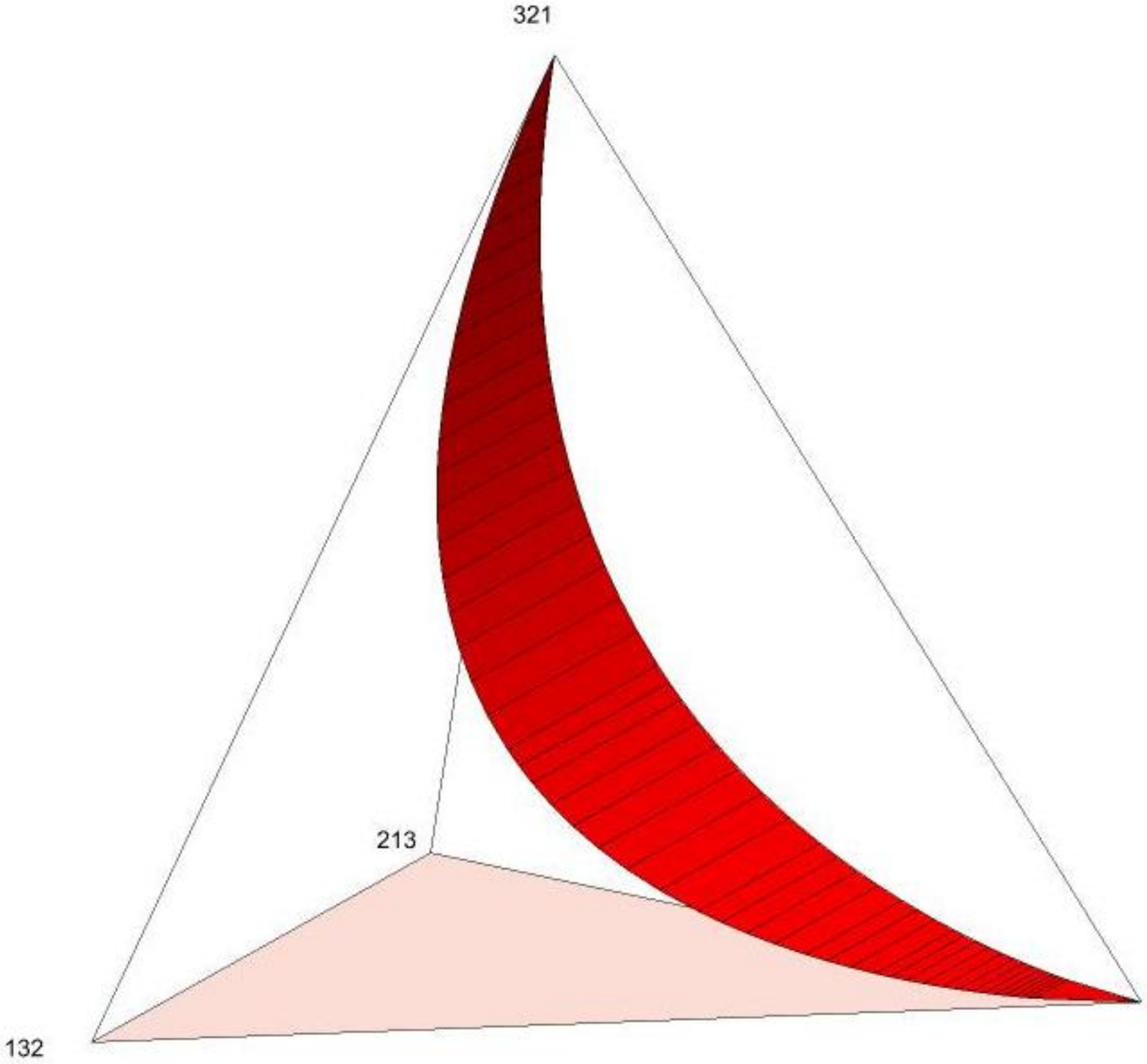
The top surface: Just the easy cross sections

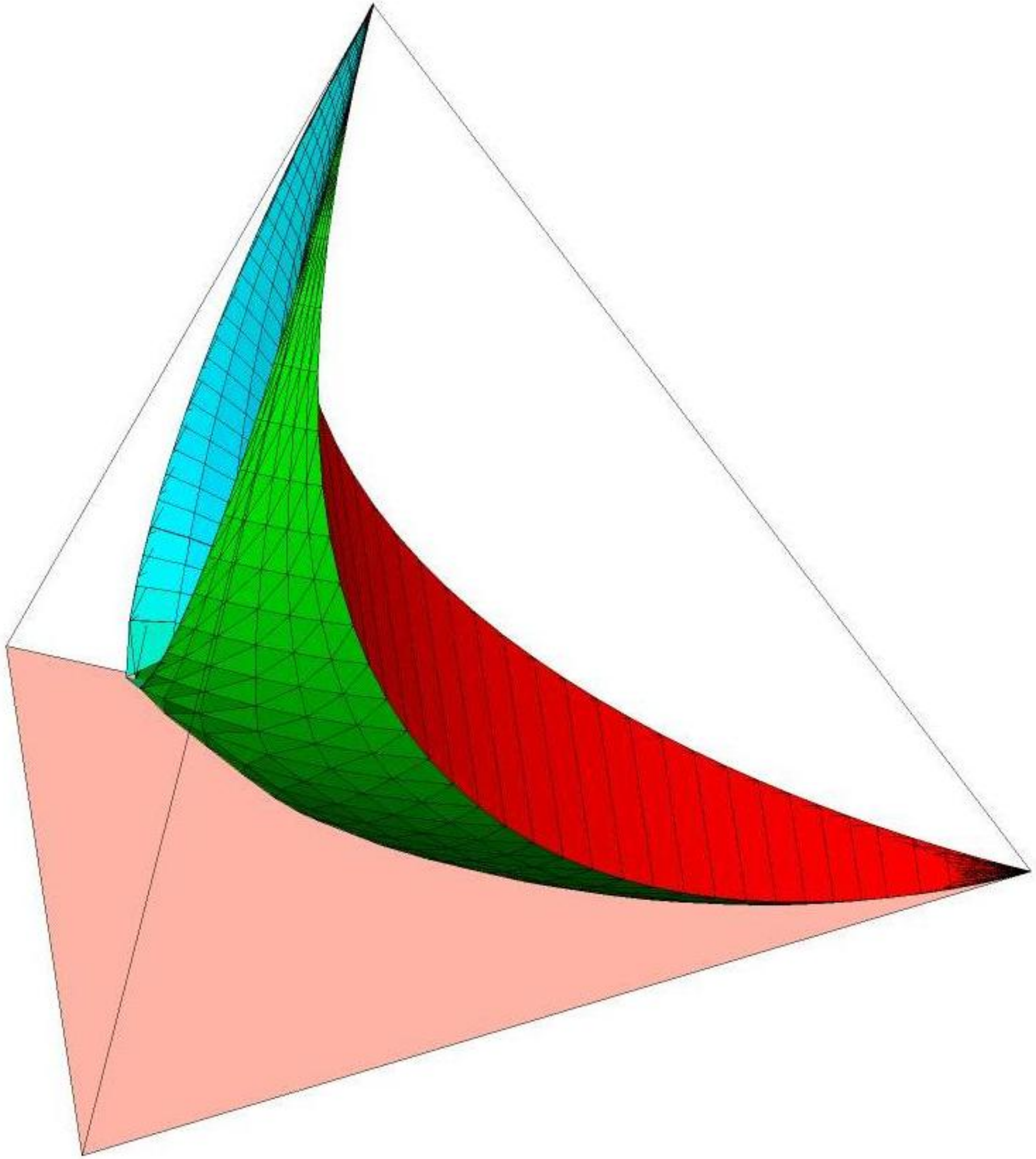


Top and Sides

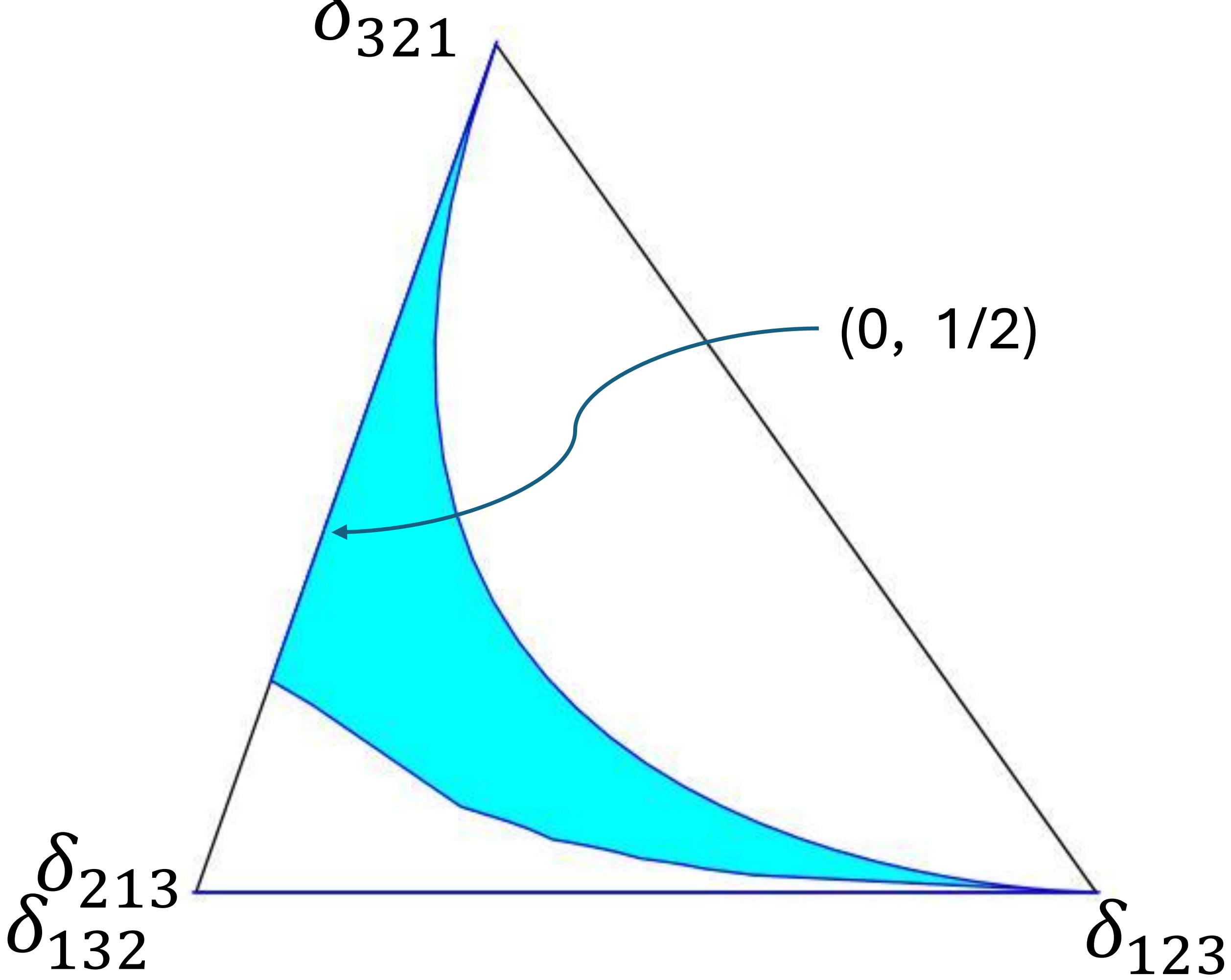


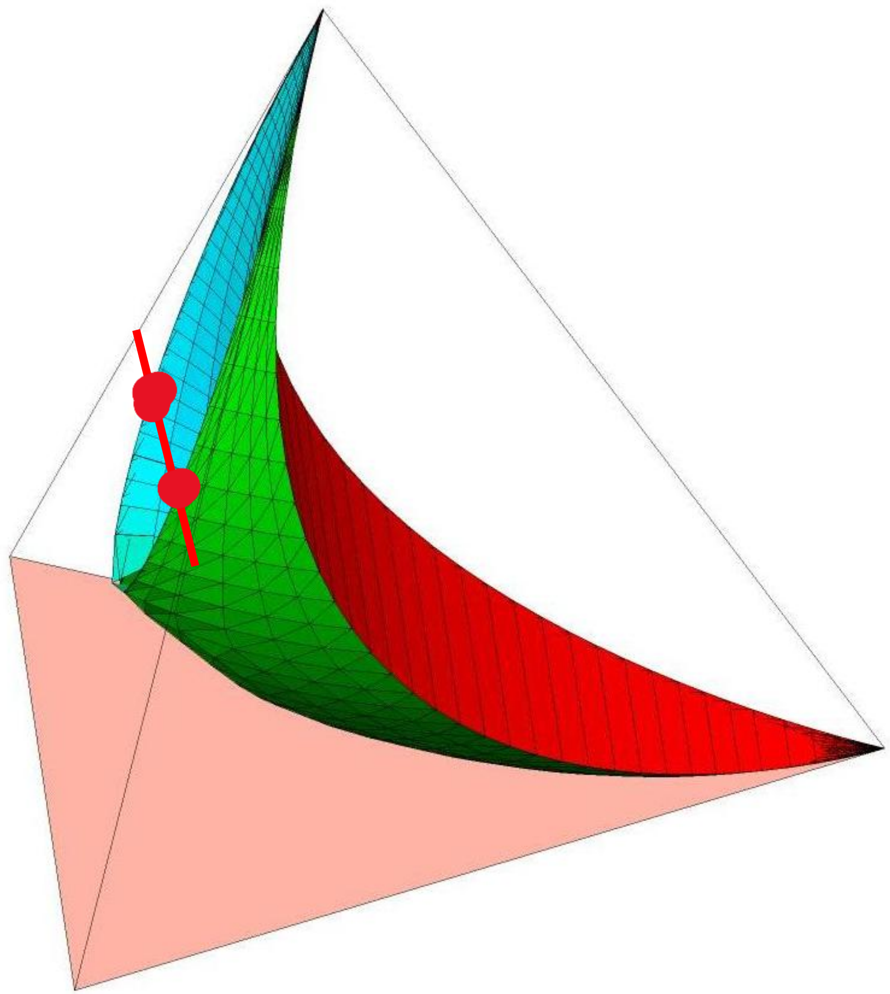



321
213
132

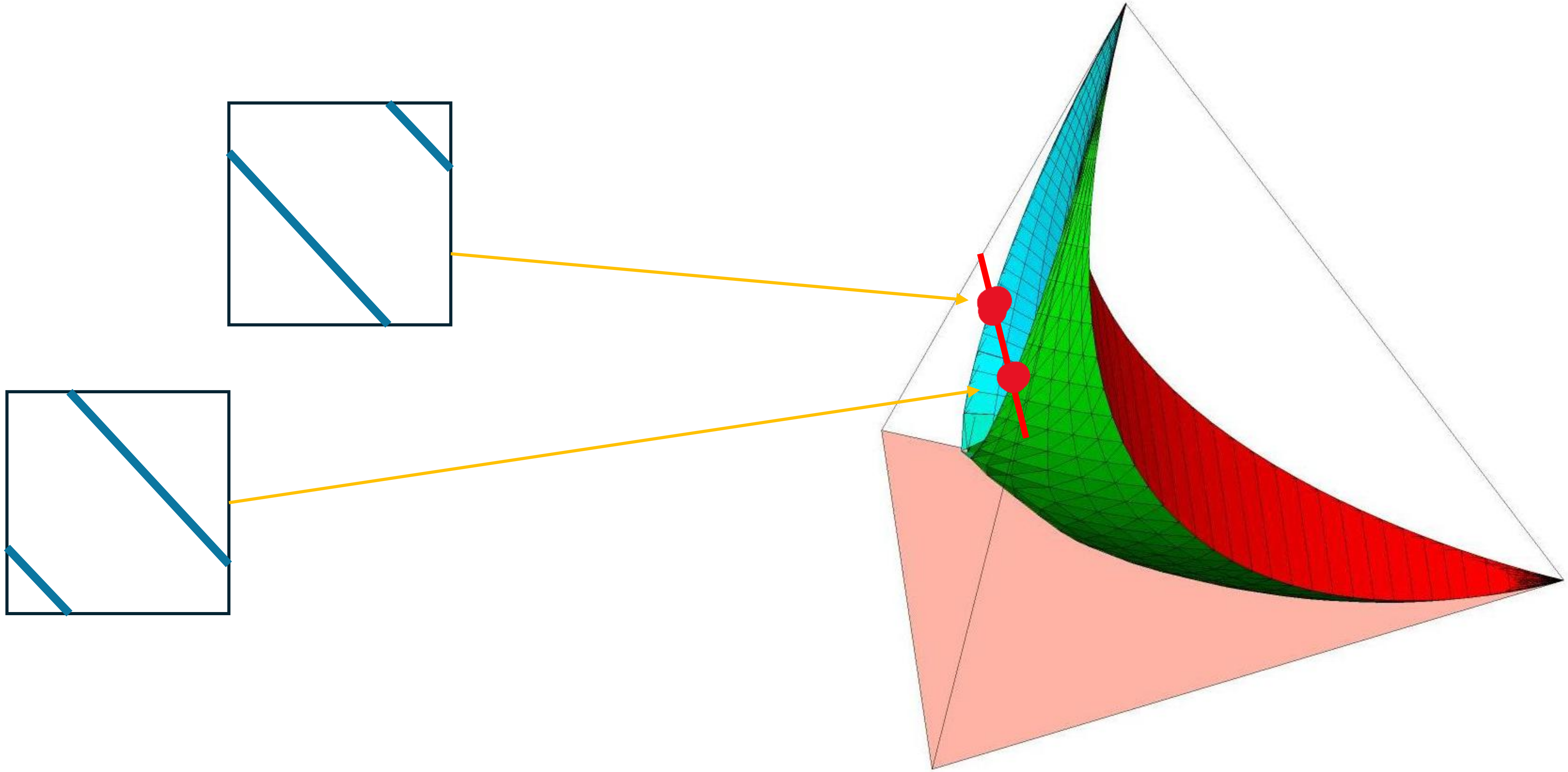


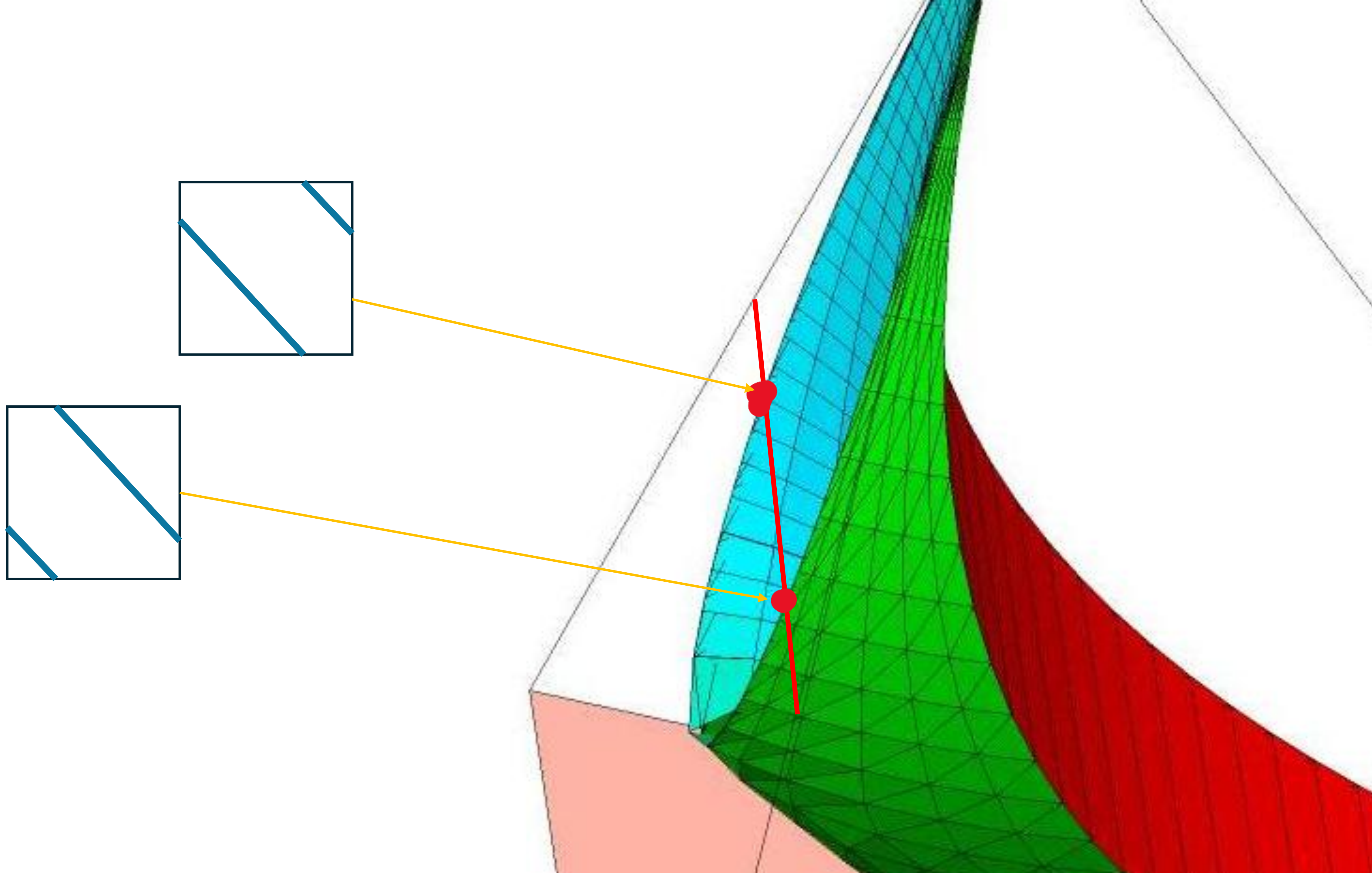
A harder cross section



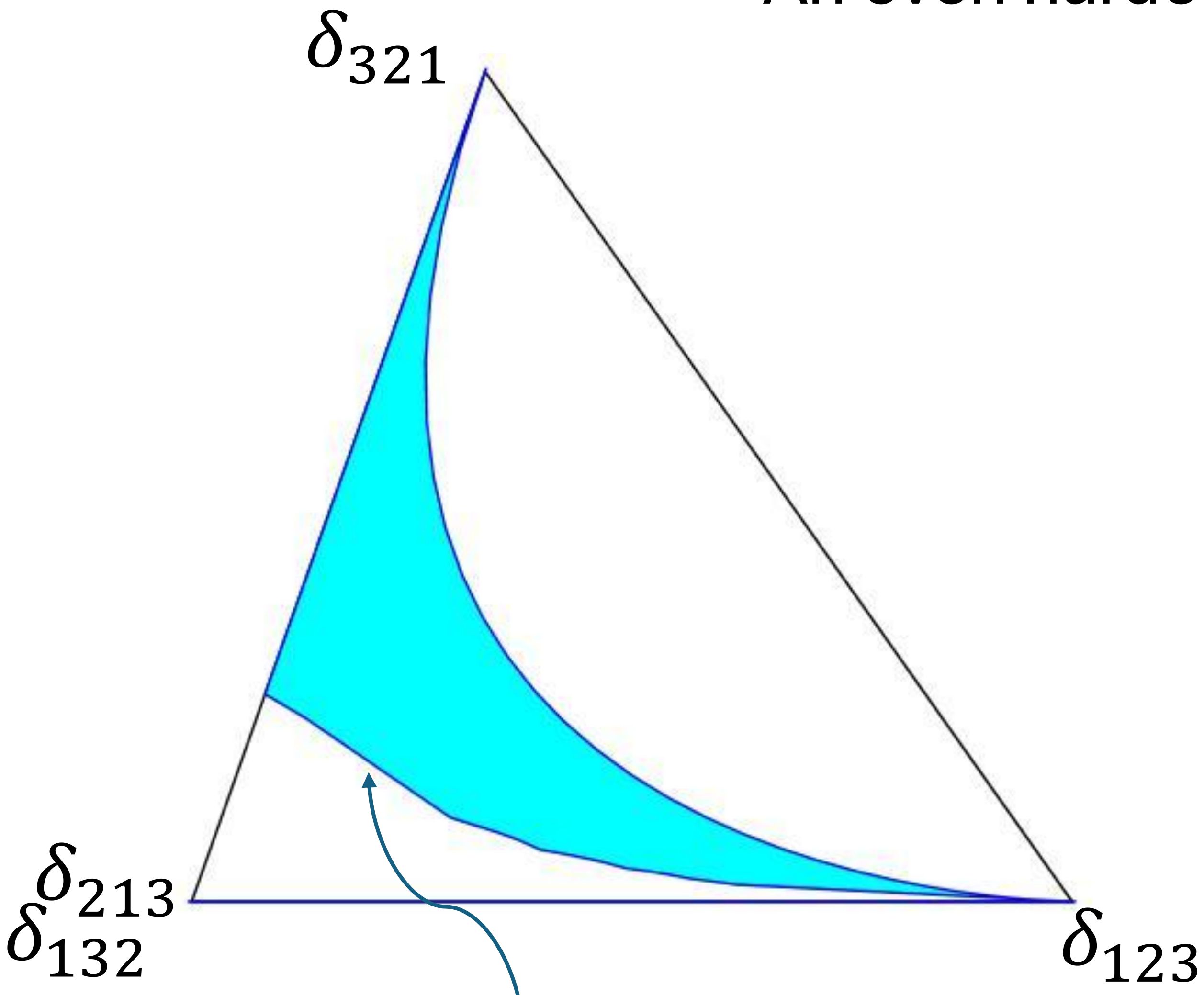


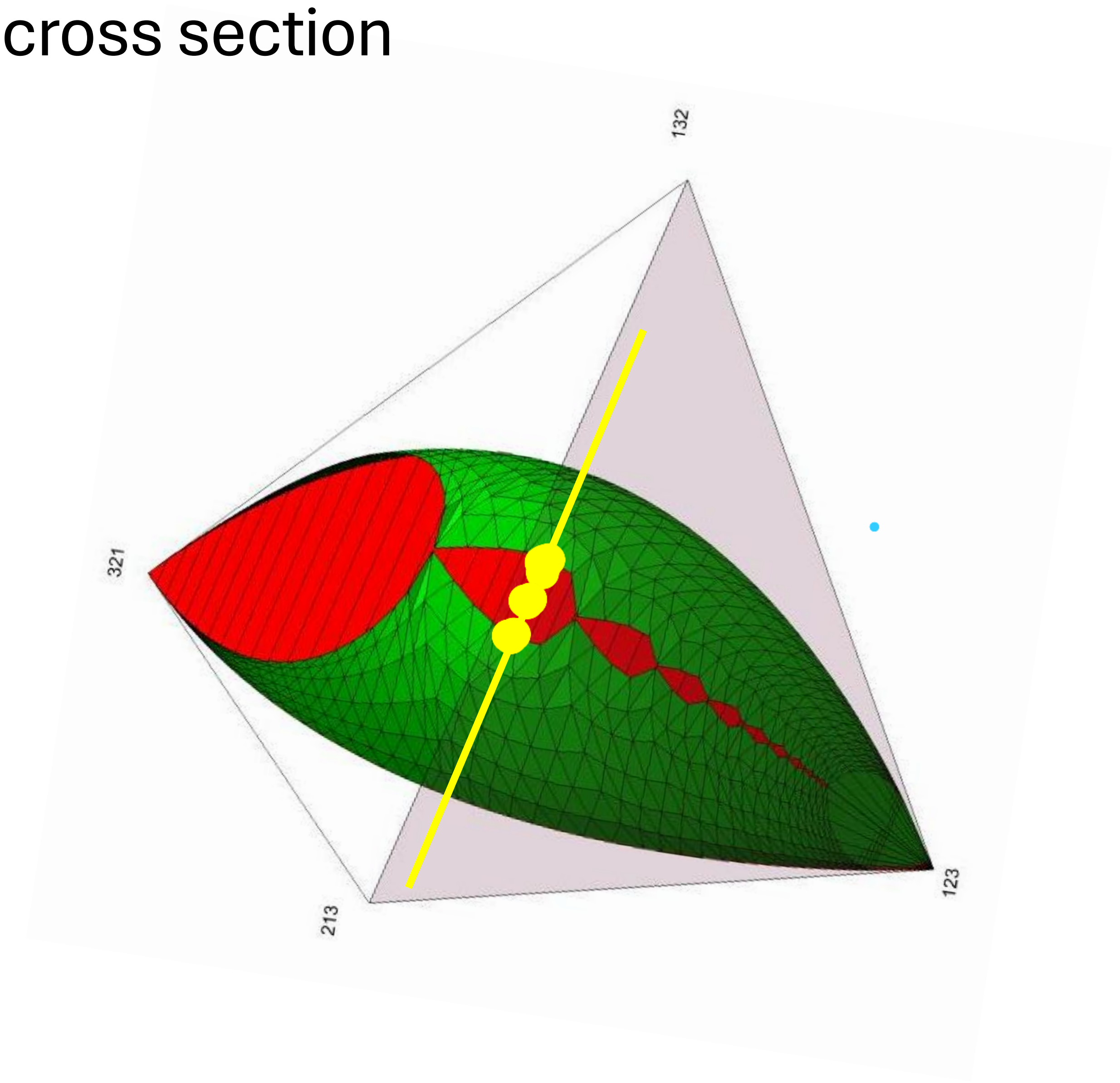
A harder cross section

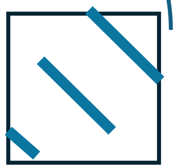




An even harder cross section

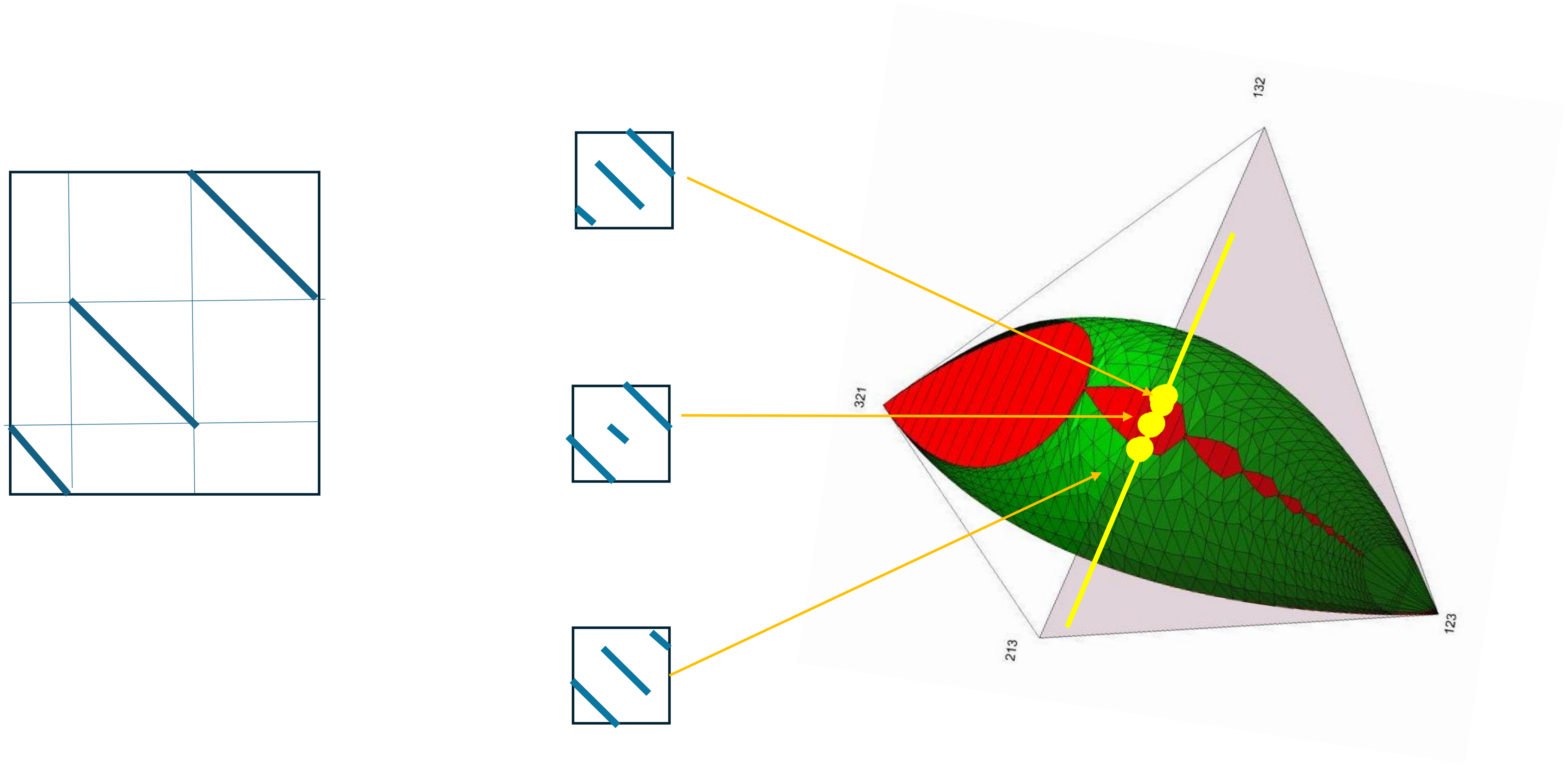




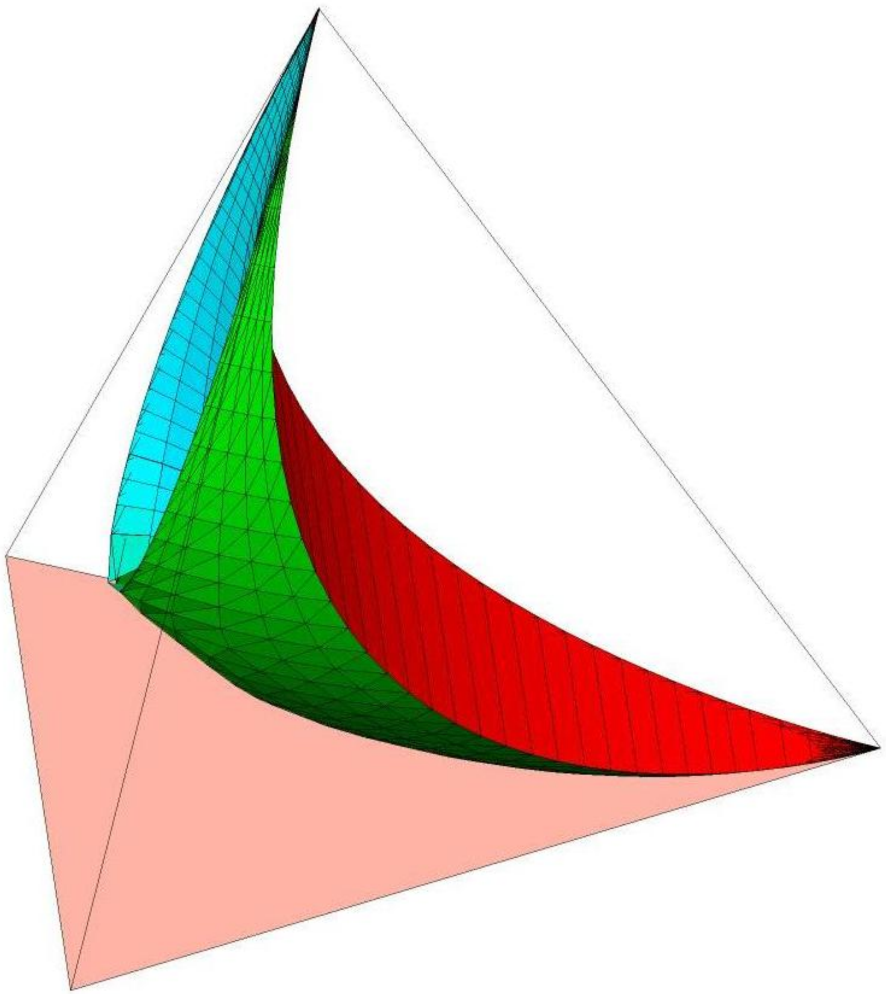


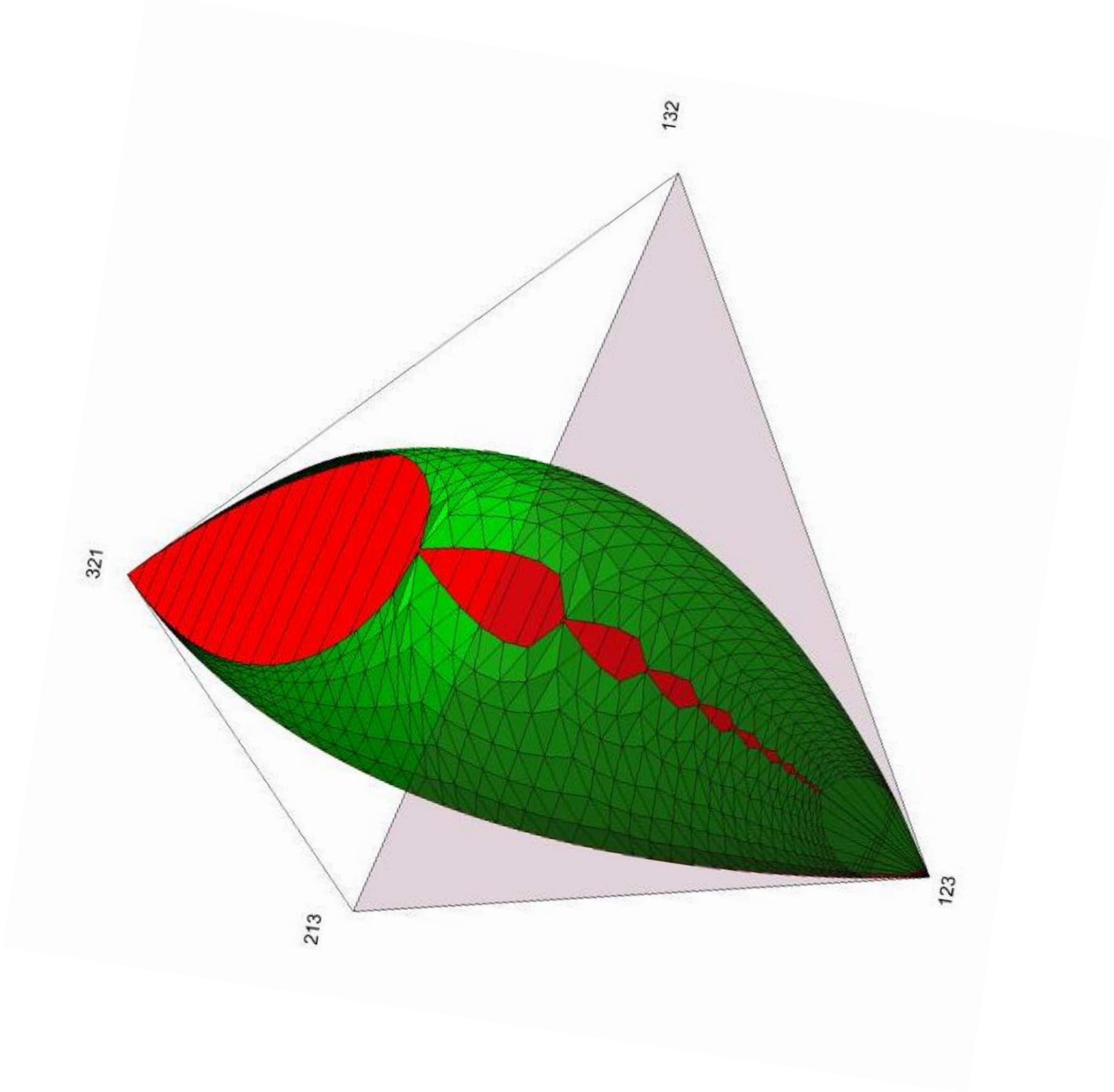
(.192, .136)

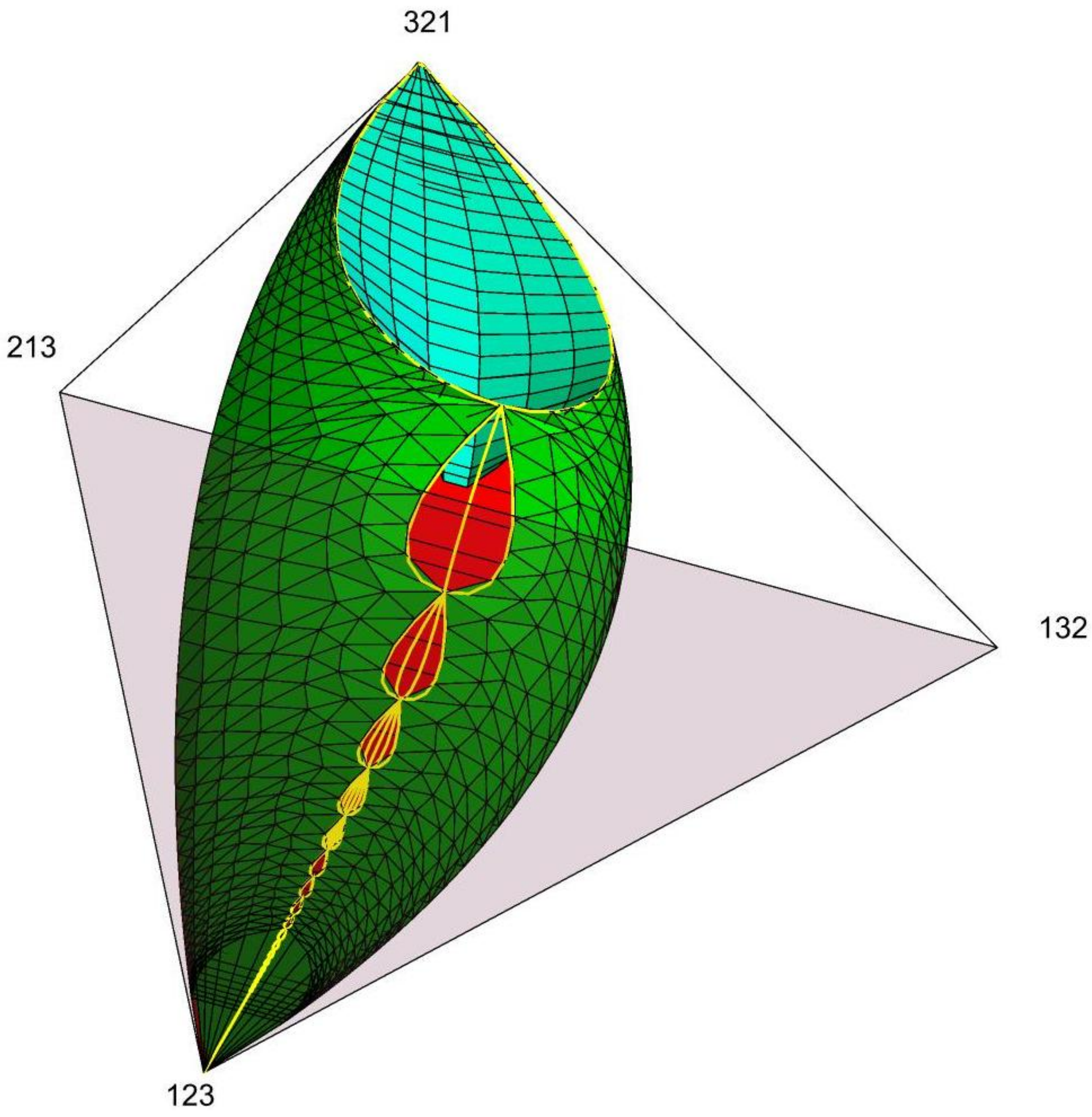
$$\delta_{123} = .192, \delta_{321} = .136$$

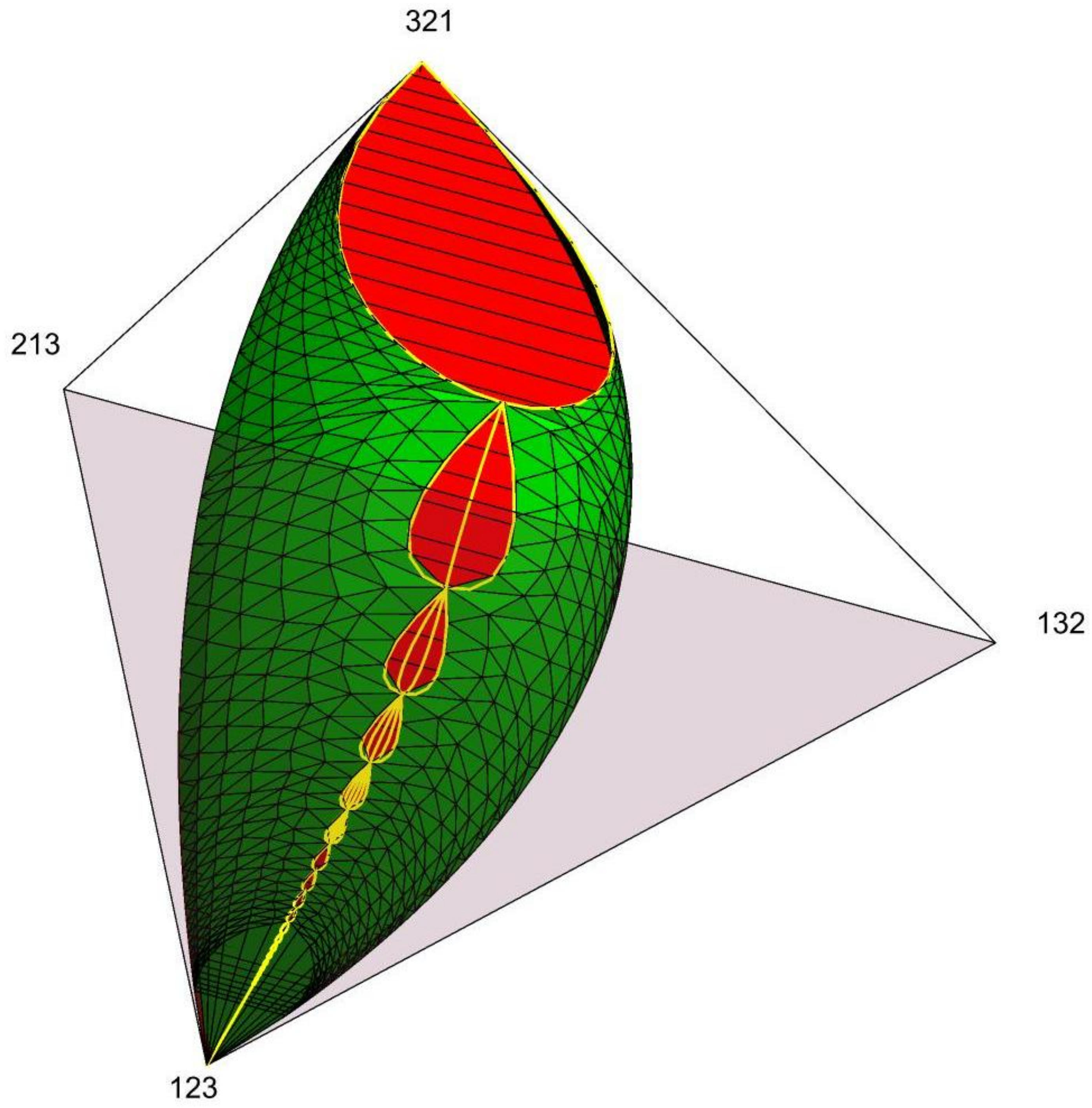

132
321
213
123

Flip it over...






321
213
132
123


321
213
132
123