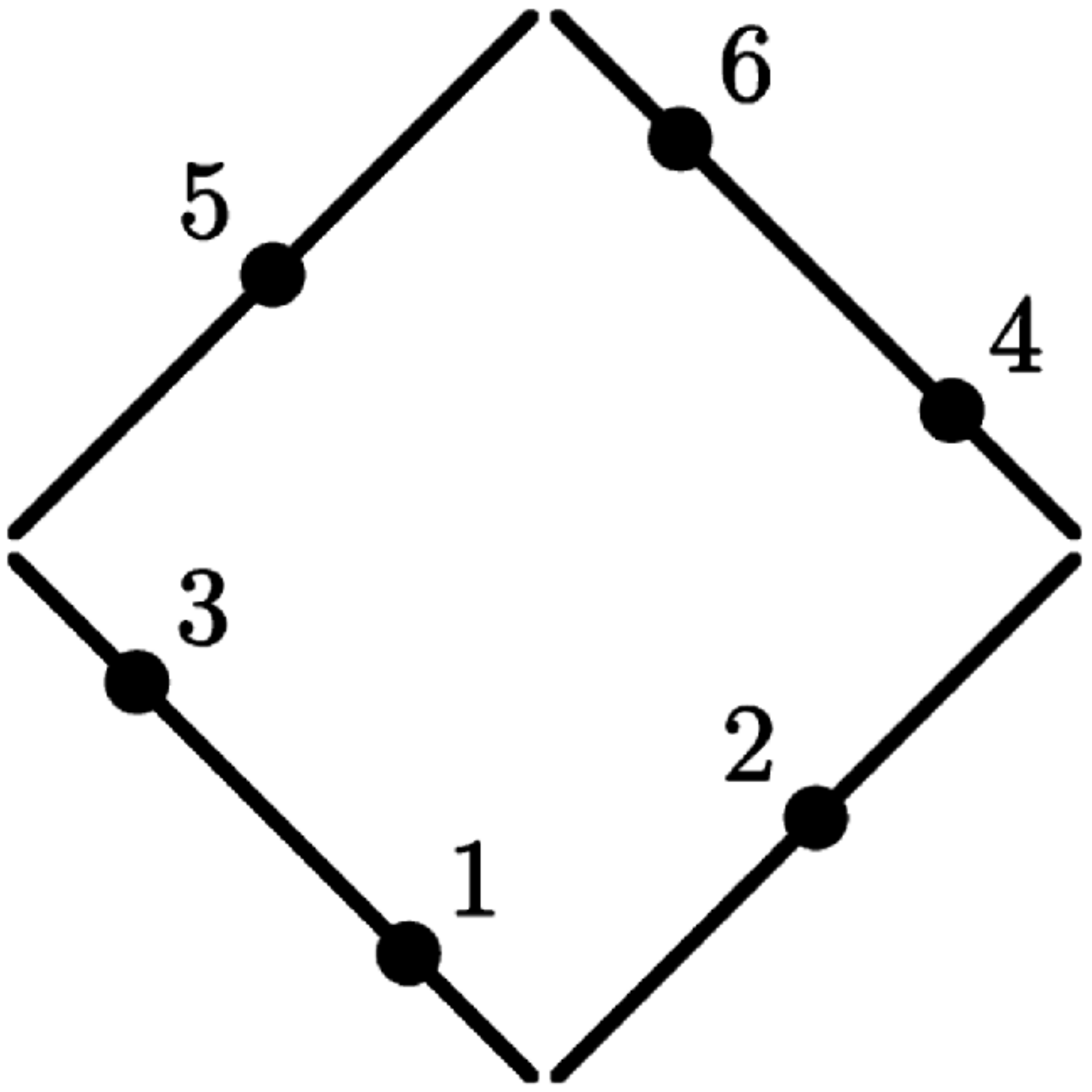
Geometric grid classes and lettericity

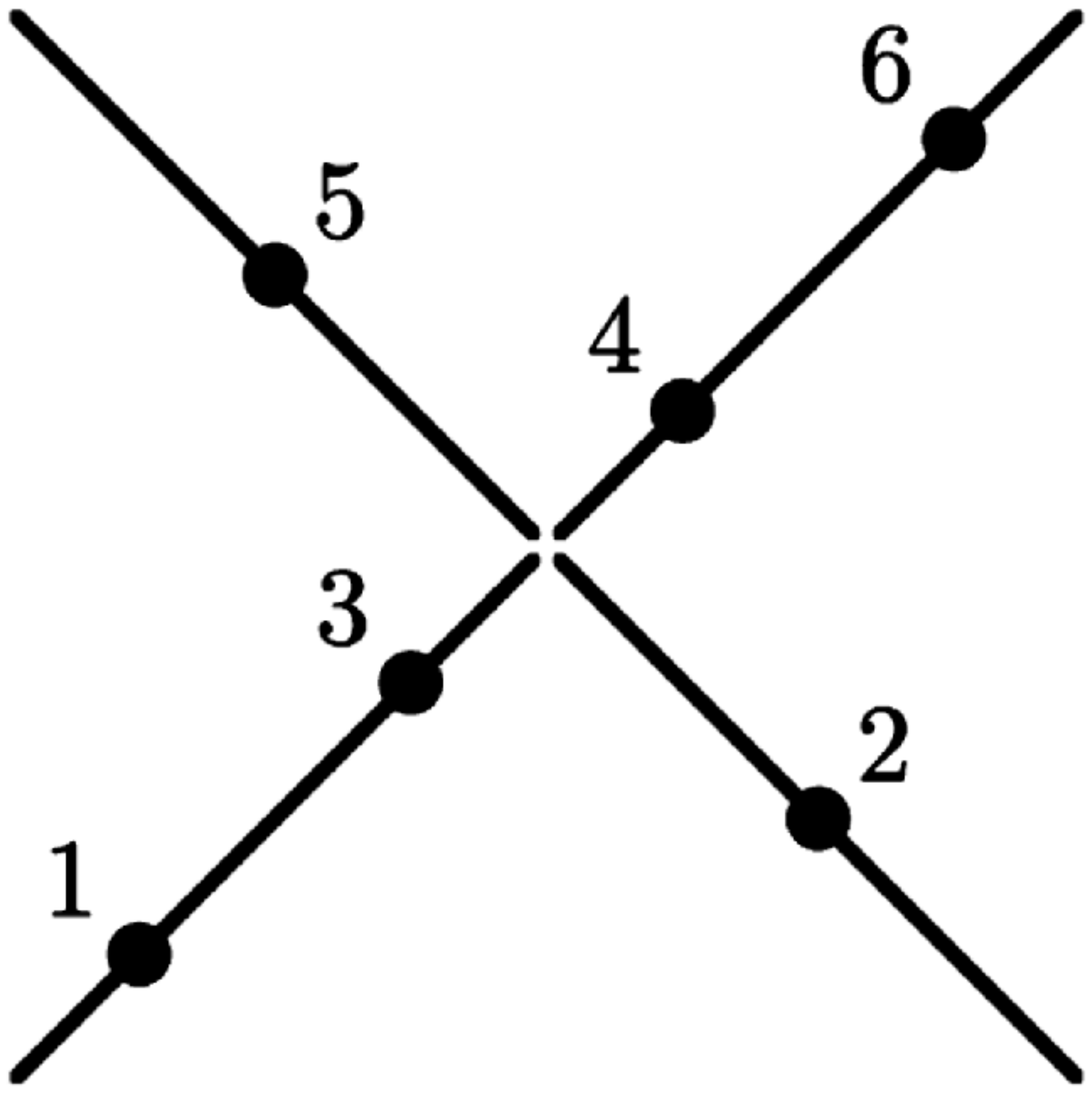
Vince Vatter
University of Florida

Permutation Patterns 2025

Geometric grid classes

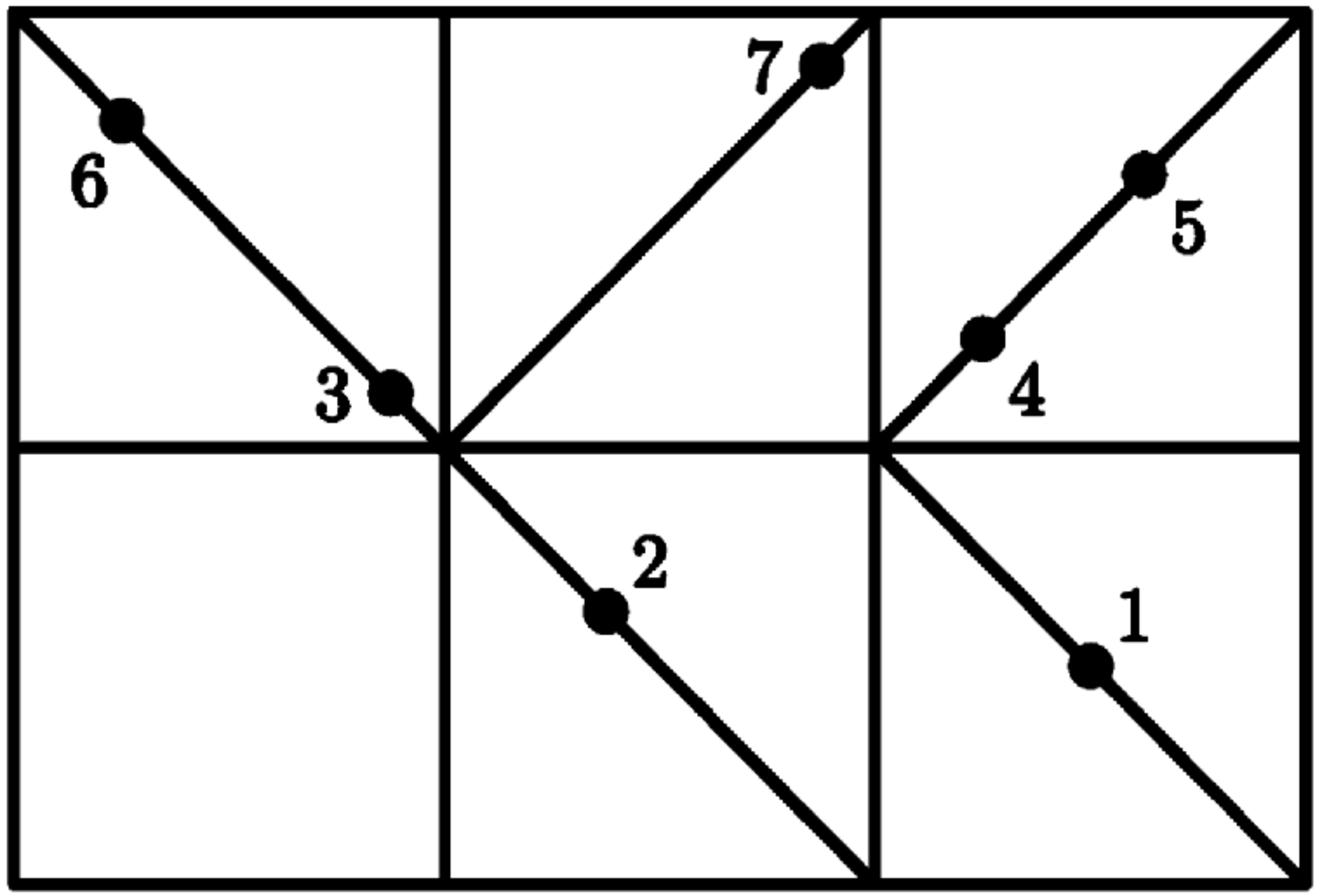
Defined by example





Geometric grid classes

Defined by example



$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

Geometric grid classes

Defined by example

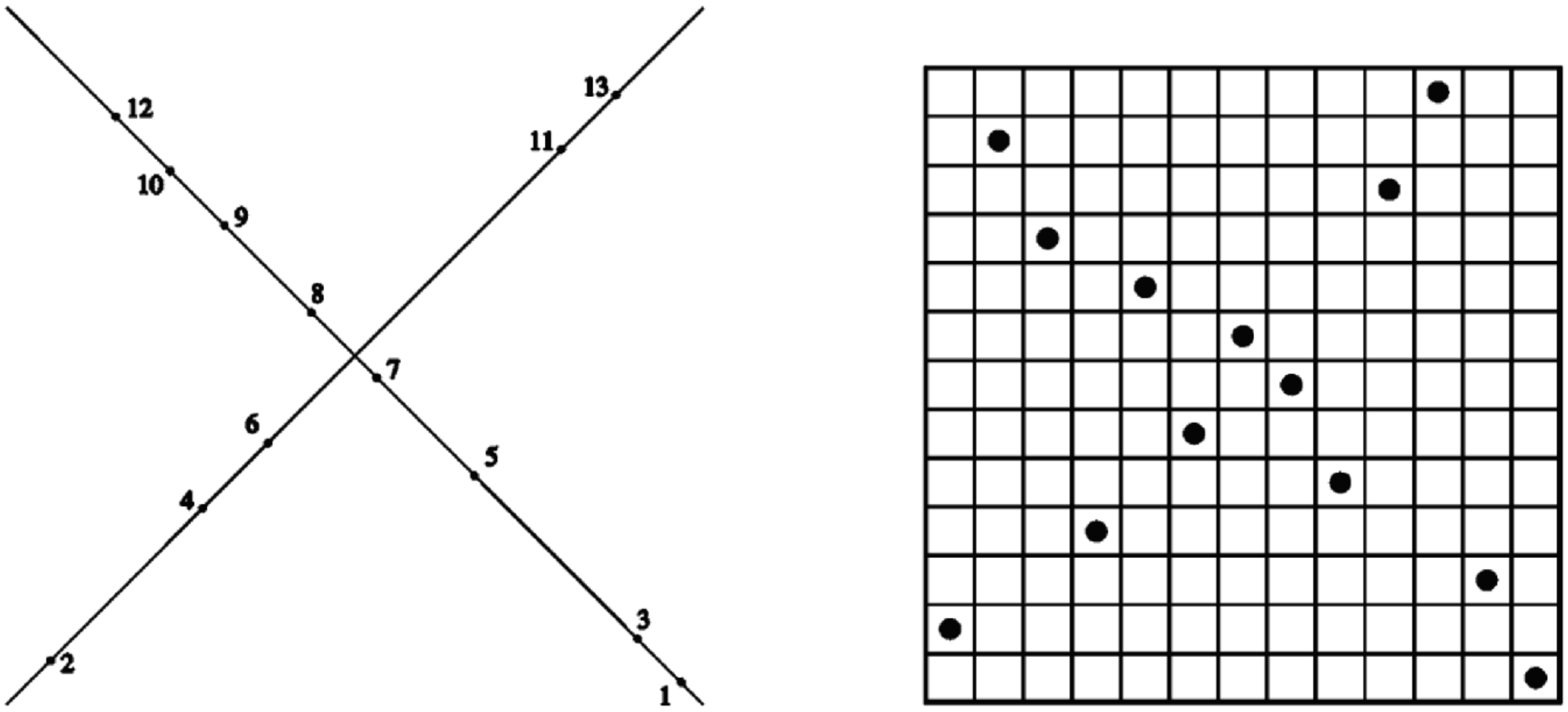
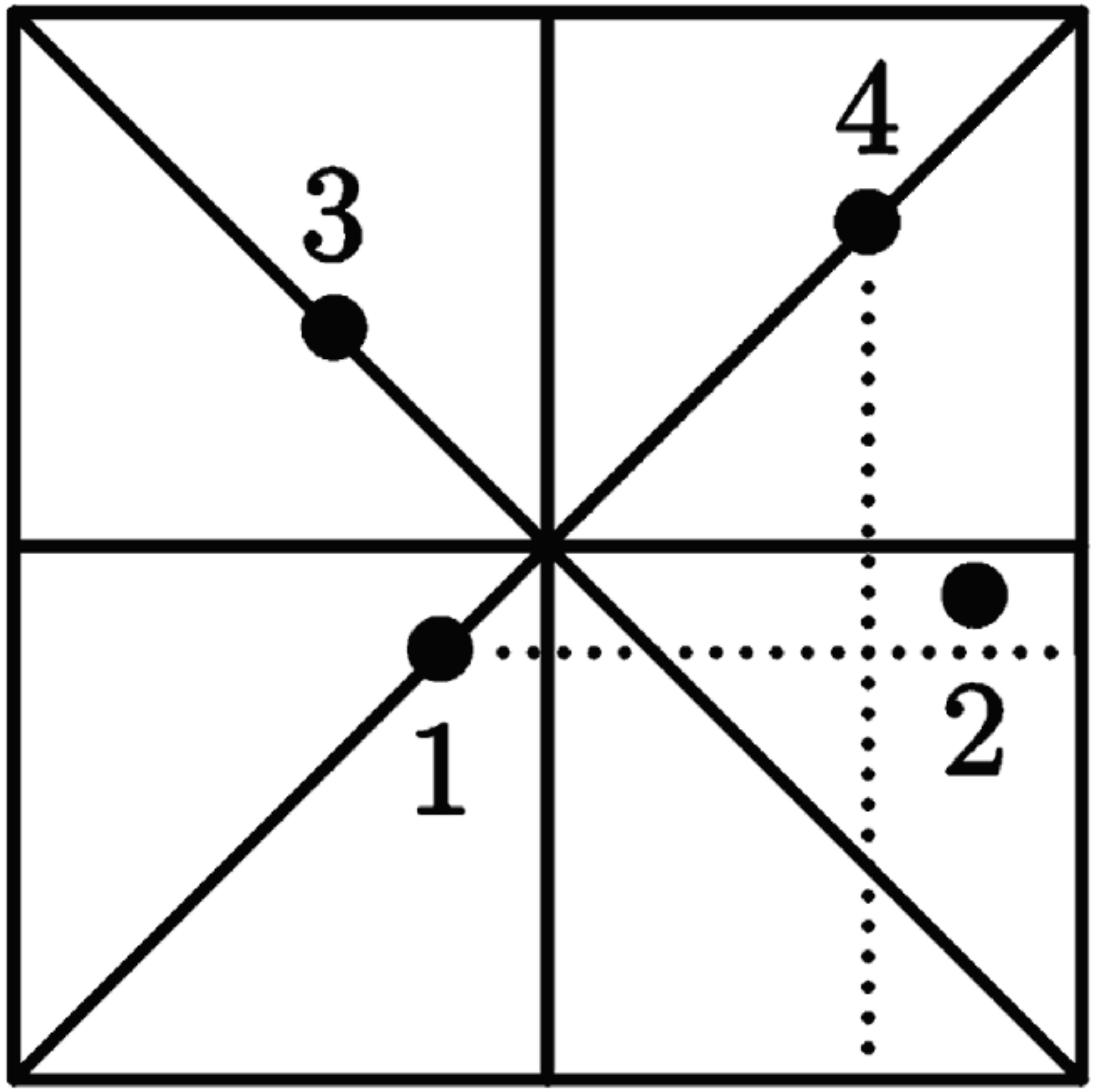


Figure 2: A permutation in the X-class, $\pi = (2, 12, 10, 4, 9, 6, 8, 7, 5, 11, 13, 3, 1)$.

Geometric grid classes

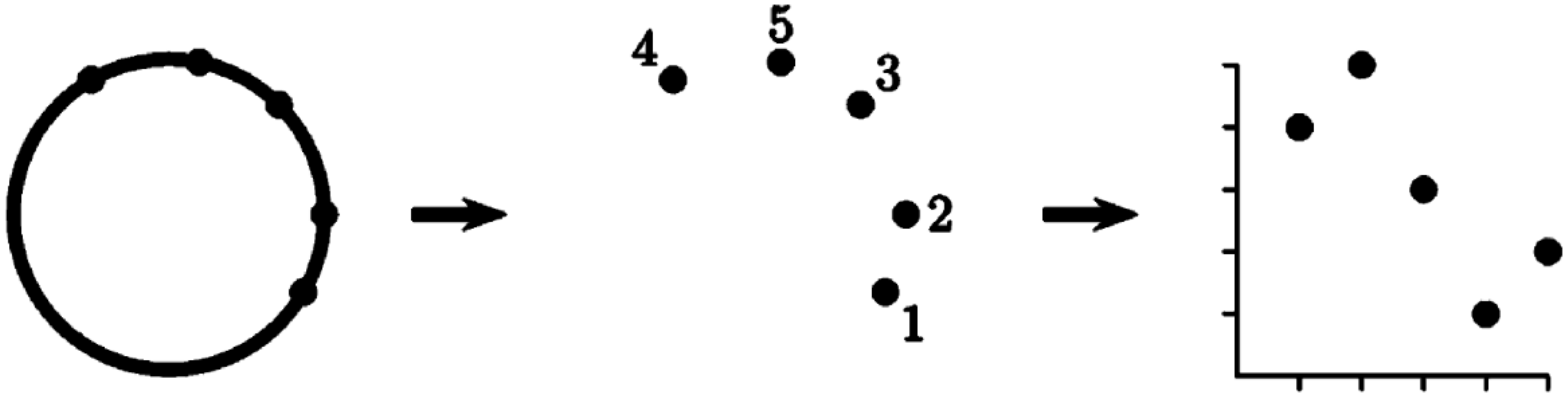
Defined by example



$$\notin \mathrm{Geom}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Geometric grid classes

Defined by example



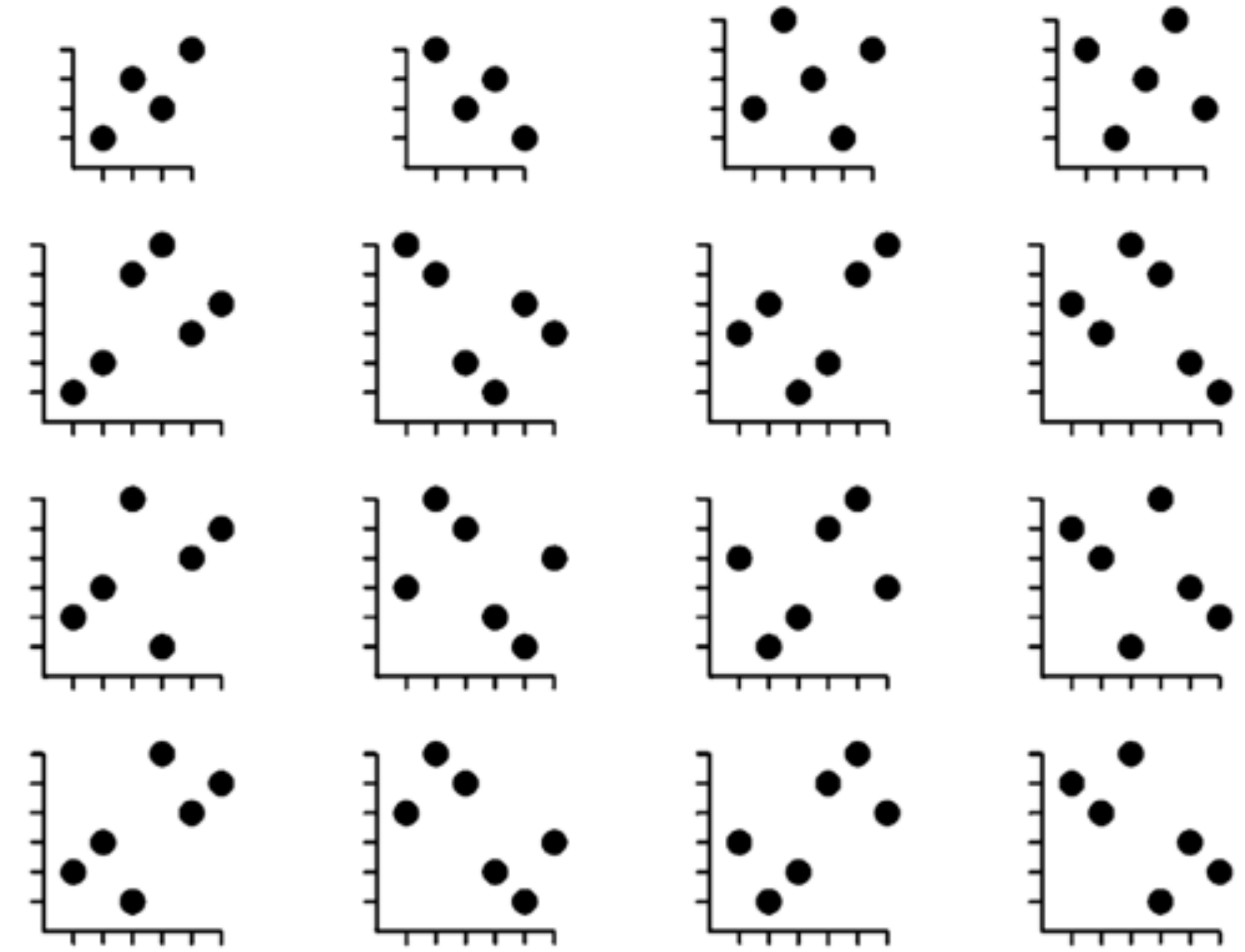
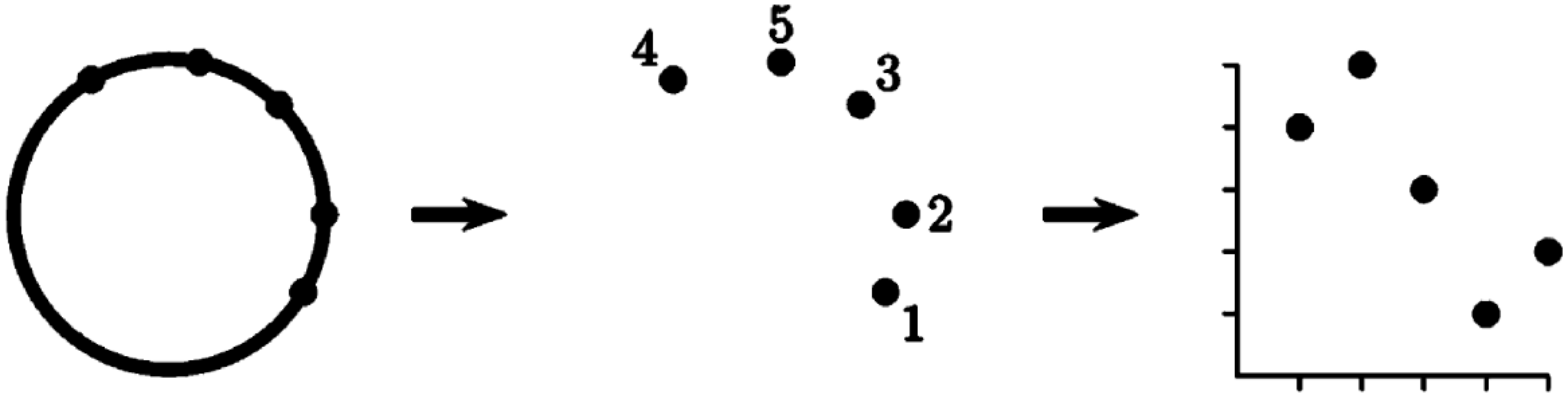


Figure 2: The basis for the class of permutations that can be drawn on a circle.

Geometric grid classes

Defined by example

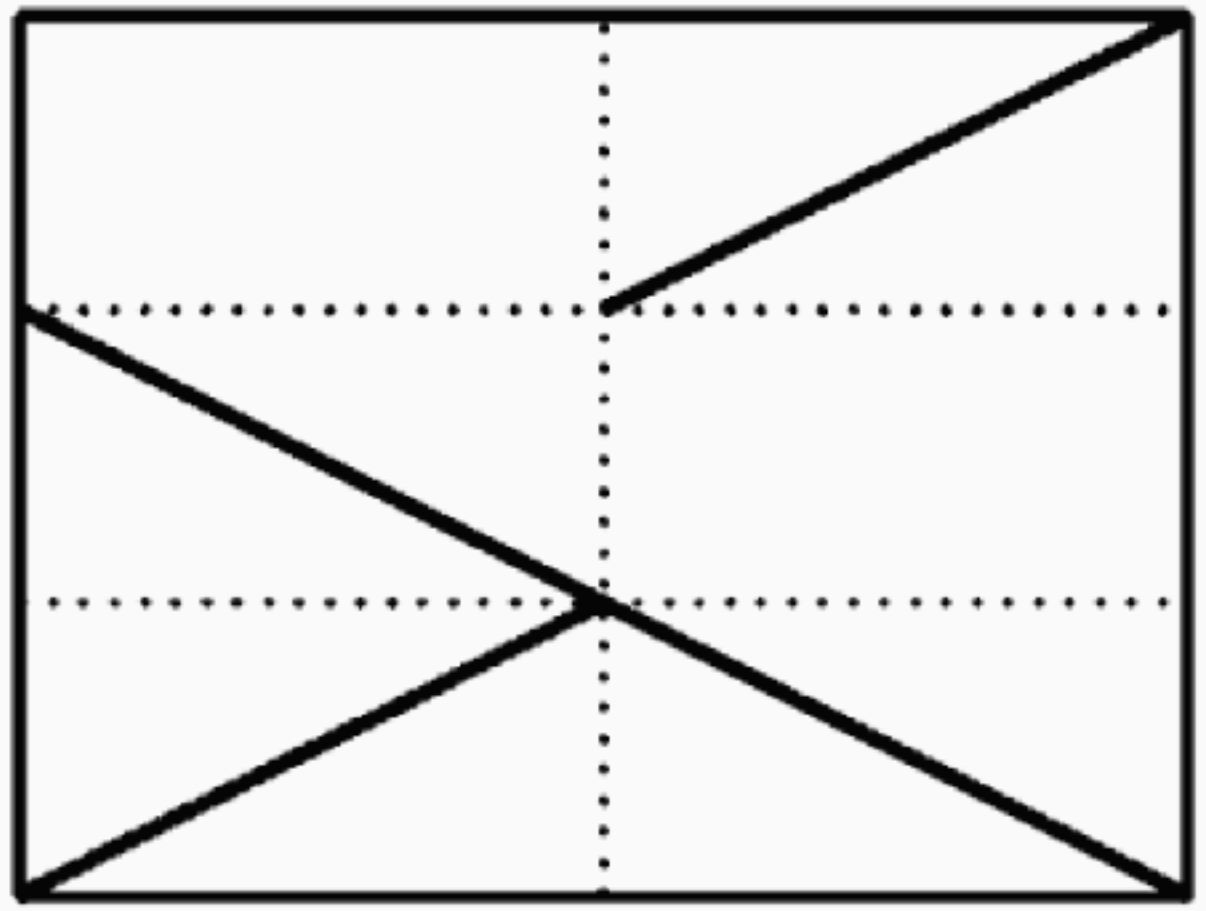


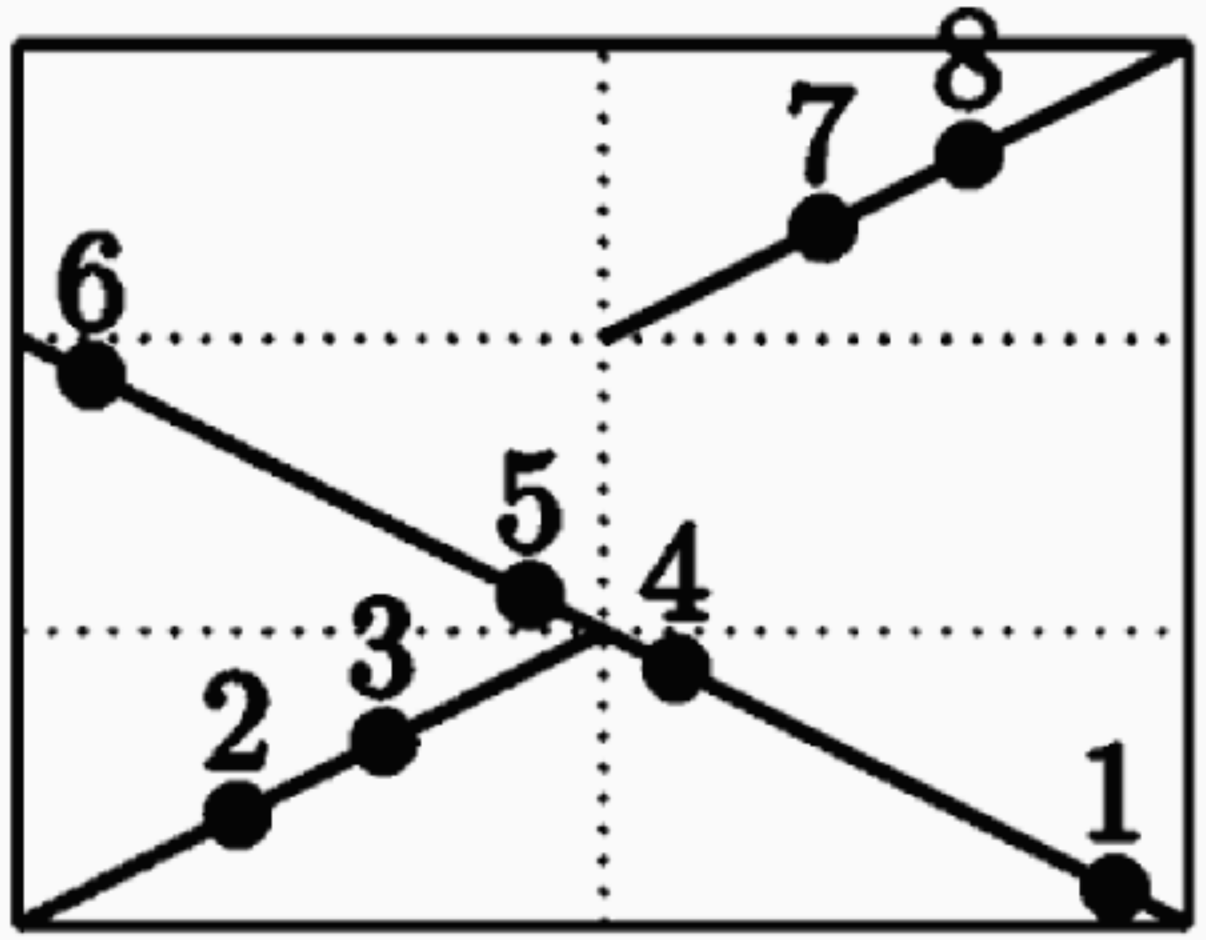
$$\frac{1-6x+12x^2-10x^3+5x^4+2x^5-2x^6}{(1-4x+2x^2)(1-x)^3}$$

Geometric grid classes

Defined by example

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$$





Theorem. *Every geometrically griddable class is well-quasi-ordered.*
(No infinite antichains.)

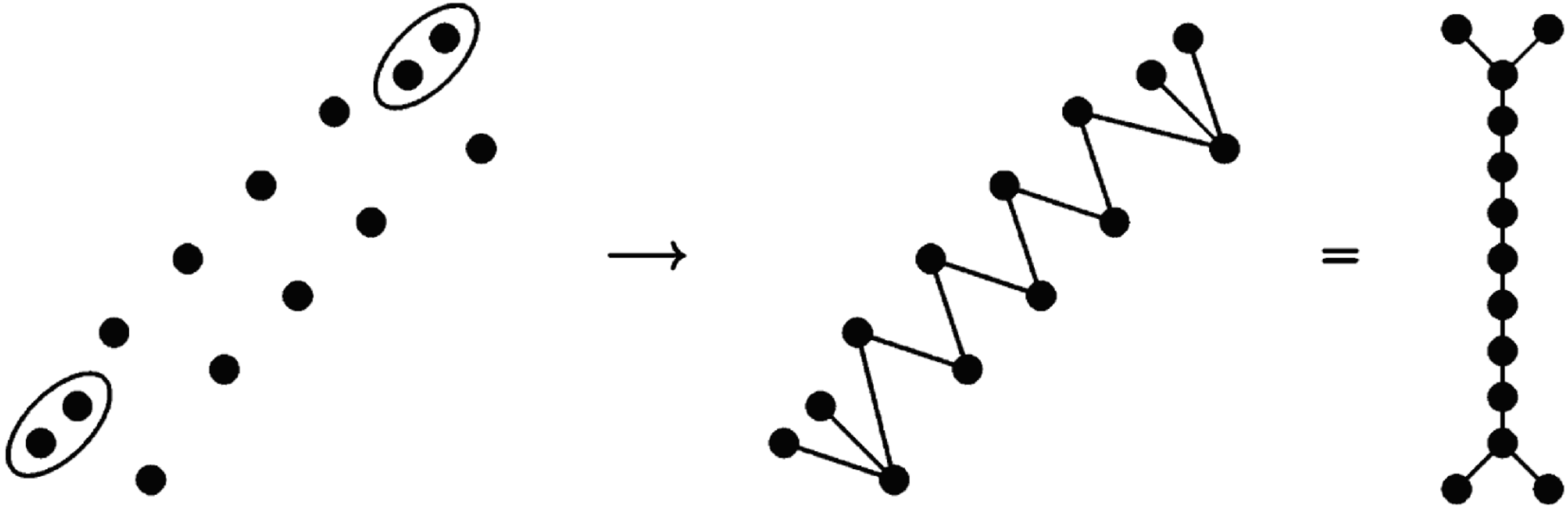
Theorem. *Every geometrically griddable class is finitely based.*
(Defined by a finite list of forbidden permutations.)

Theorem. *Every geometric grid class is in bijection with a regular language.*
(And thus has a rational generating function.)

Albert, Michael H.; Atkinson, M. D.; Bouvel, Mathilde; Ruškuc, Nik; Vatter, Vincent
Geometric grid classes of permutations.
Trans. Am. Math. Soc. 365, No. 11, 5859-5881 (2013).

Inversion graphs

(Aka permutation graphs)



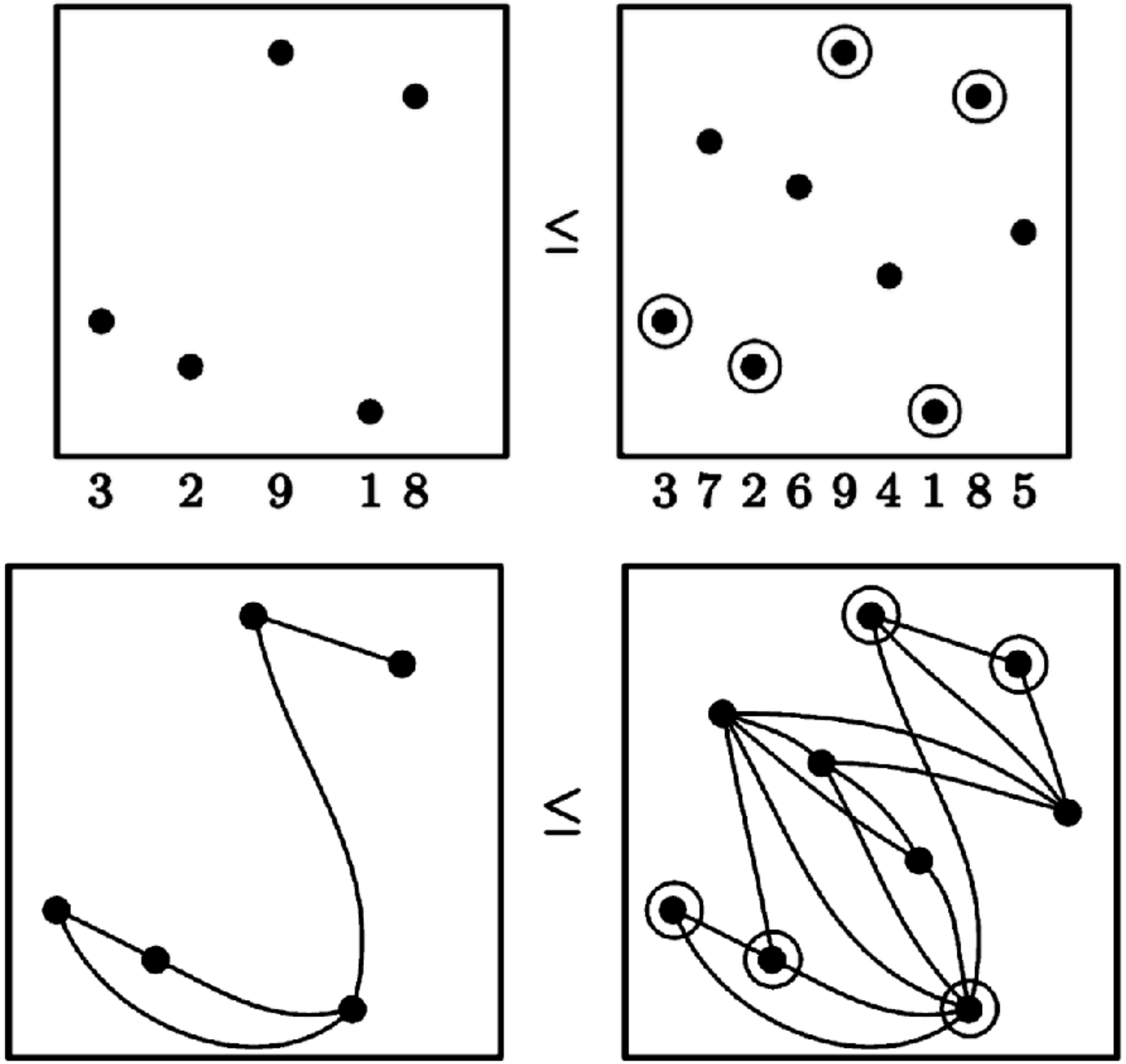


FIG. 3. *The containment order on permutations and the induced subgraph order on their corresponding inversion graphs.*

Letter graphs

Let $\mathscr{P} \subseteq \Sigma^2$ be a fixed set of ordered pairs of symbols from Σ. To each word $w = w_1 w_2 \ldots w_n$ where $w_i \in \Sigma$ we assign its *letter graph* $G(\mathscr{P}, w)$ in the following way:

$$V(G(\mathscr{P}, w)) = \{1, 2, \ldots, n\},$$

$$E(G(\mathscr{P}, w)) = \{\{i, j\}; w_{\min(i,j)} w_{\max(i,j)} \in \mathscr{P}\}.$$

The vertices of $G(\mathscr{P}, w)$ are naturally labelled with the symbols of w.

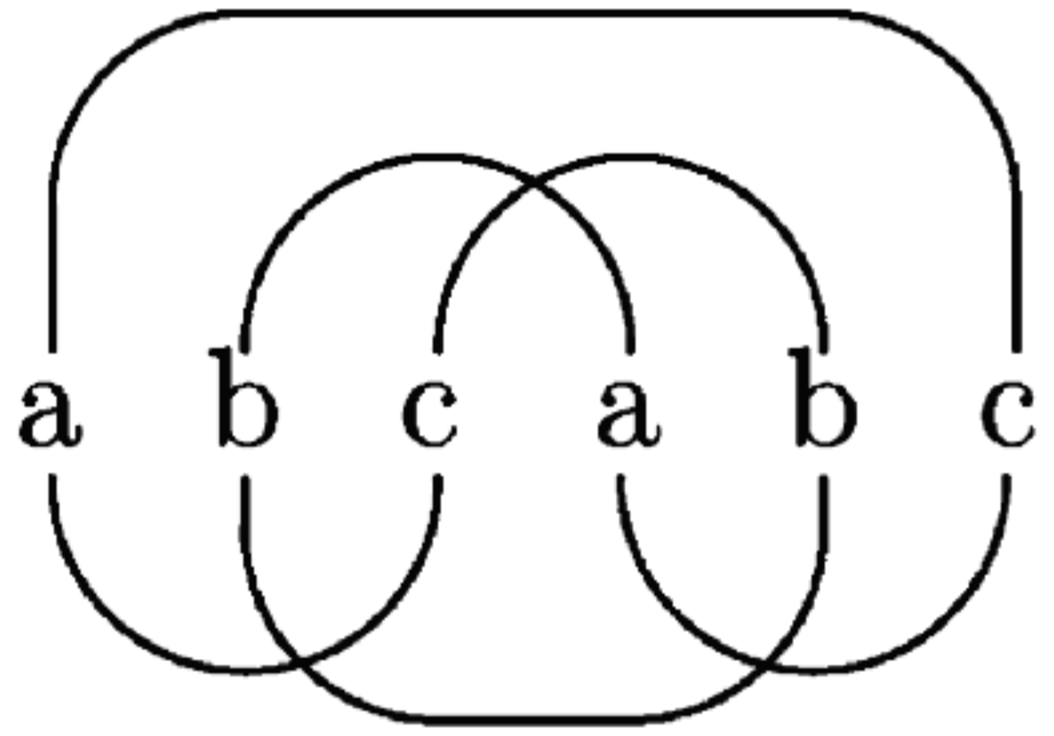


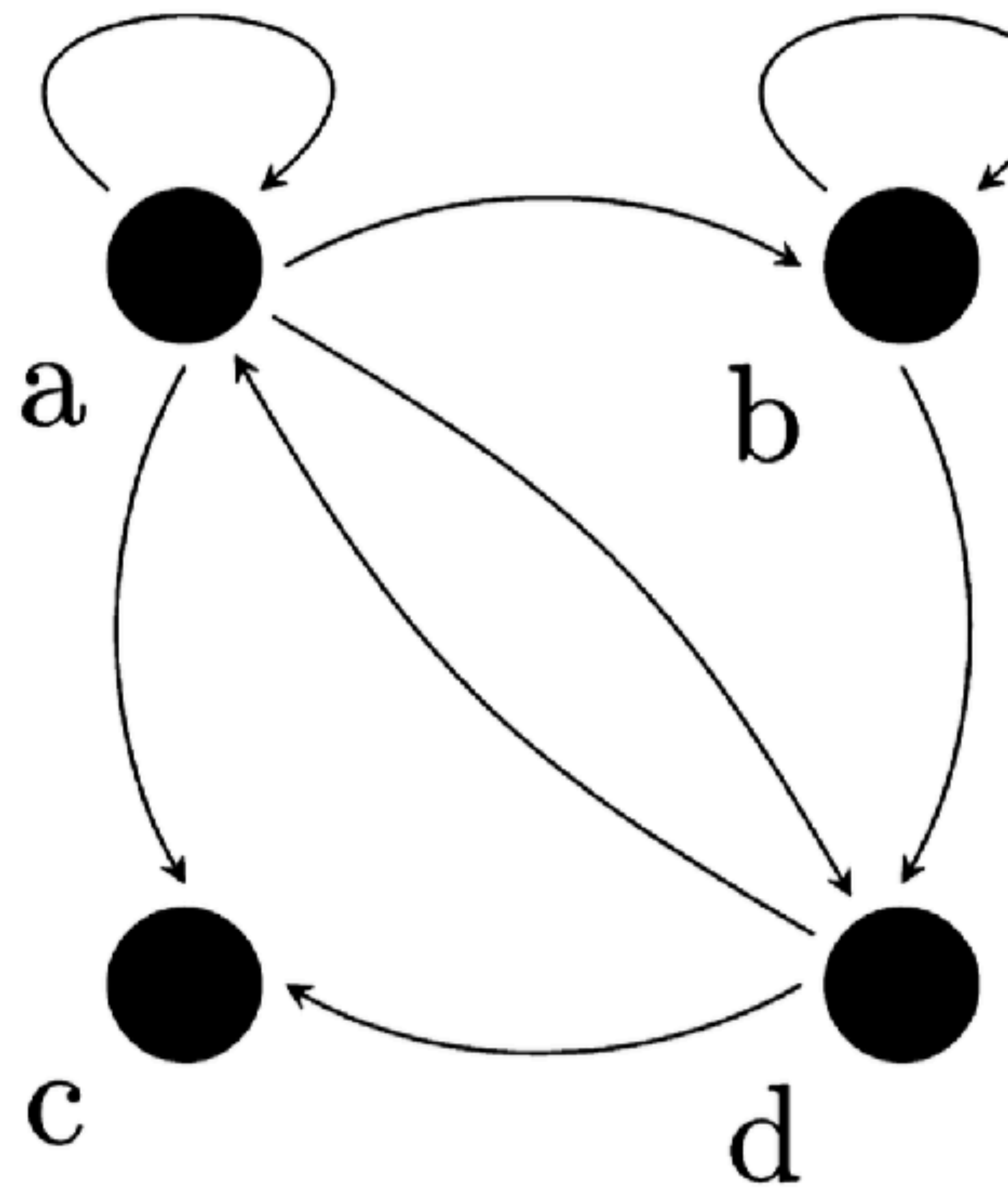
Fig. 1. C_6 as a 3-letter graph ($\mathscr{P} = \{ac, ba, cb, bb\}$).

Petkovšek, Marko

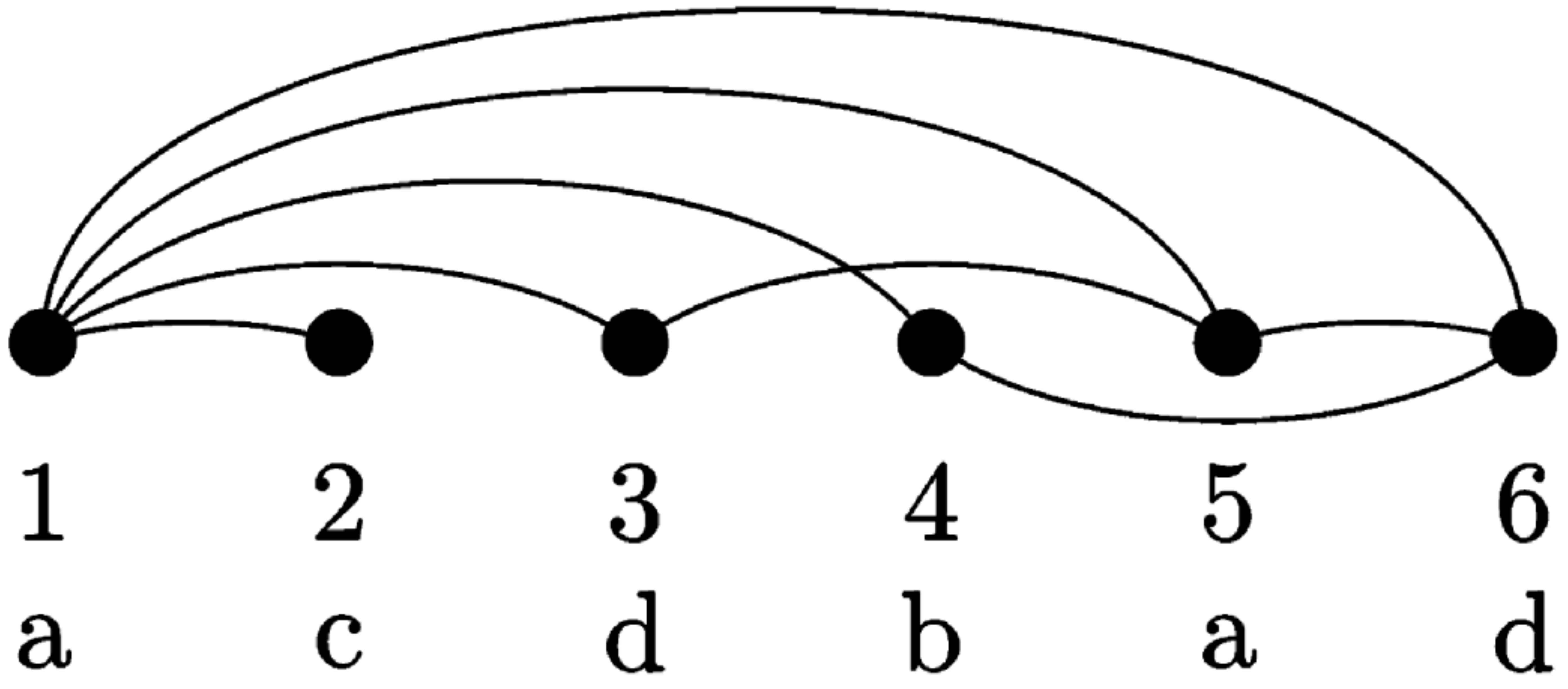
Letter graphs and well-quasi-order by induced subgraphs.

Discrete Math. 244, No. 1-3, 375-388 (2002).

Letter graphs



(a) The decoder $\mathcal{D}$



(b) Letter graph $G(\mathcal{D}, acdbad)$

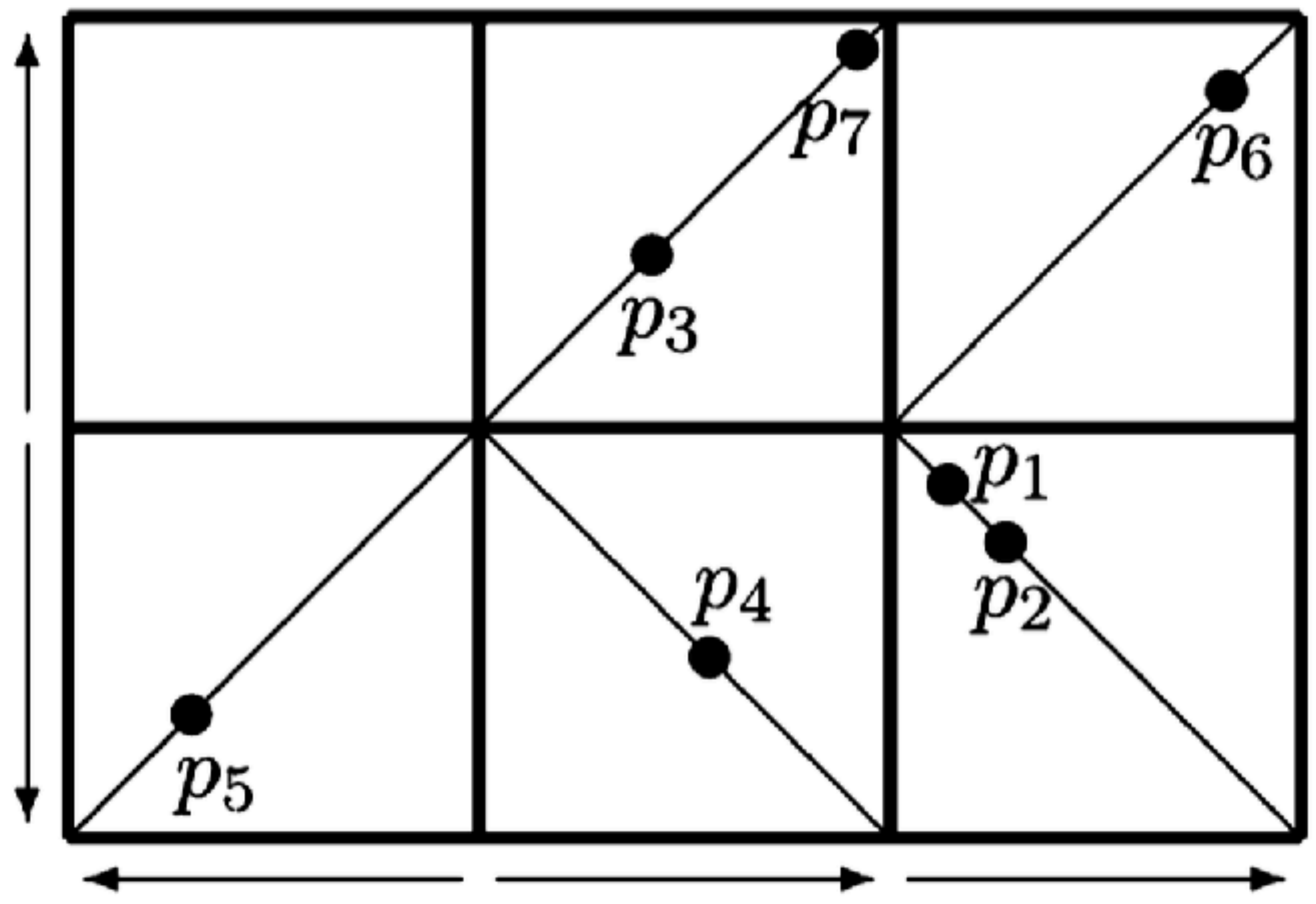
Theorem. For $n \geq 3$, the lettericity of P_n is $\lfloor \frac{n+4}{3} \rfloor$

Ferguson, Robert

On the lettericity of paths.

Australas. J. Comb. 78, Part 2, 348-351 (2020).

Conjecture 1. *Let X be a class of permutations and $\mathcal{G}_X$ the corresponding class of permutation graphs. Then X is geometrically griddable if and only if $\mathcal{G}_X$ is a class of k-letter graphs for a finite value of k.*



- $M_{k,\ell} = 1$, then the points lying in the cell $C_{k,\ell}$ form an independent set in the permutation graph of π. Therefore, we do not include the pair $(a_{k\ell}, a_{k\ell})$ in $\mathcal{P}$.
- $M_{k,\ell} = -1$, then the points lying in the cell $C_{k,\ell}$ form a clique in the permutation graph of π. Therefore, we include the pair $(a_{k\ell}, a_{k\ell})$ in $\mathcal{P}$.
- two cells $C_{k,\ell}$ and $C_{s,t}$ are independent with $k < s$ and $\ell < t$, then no point of $C_{k,\ell}$ is adjacent to any point of $C_{s,t}$ in the permutation graph of π. Therefore, we include neither $(a_{k\ell}, a_{st})$ nor $(a_{st}, a_{k\ell})$ in $\mathcal{P}$.
- two cells $C_{k,\ell}$ and $C_{s,t}$ are independent with $k < s$ and $\ell > t$, then every point of $C_{k,\ell}$ is adjacent to every point of $C_{s,t}$ in the permutation graph of π. Therefore, we include both pairs $(a_{k\ell}, a_{st})$ and $(a_{st}, a_{k\ell})$ in $\mathcal{P}$.
- two cells $C_{k,\ell}$ and $C_{s,t}$ share a column, i.e., $k = s$, then we look at the sign (direction) associated with this column and the relative position of the two cells within the column.
  - If $c_k = 1$ (i.e., the column is oriented from left to right) and $\ell > t$ (the first of the two cells is above the second one), then only the pair $(a_{k\ell}, a_{kt})$ is included in $\mathcal{P}$.
  - If $c_k = 1$ and $\ell < t$, then only the pair $(a_{kt}, a_{k\ell})$ is included in $\mathcal{P}$.
  - If $c_k = -1$ (i.e., the column is oriented from right to left) and $\ell > t$ (the first of the two cells is above the second one), then only the pair $(a_{kt}, a_{k\ell})$ is included in $\mathcal{P}$.
  - If $c_k = -1$ and $\ell < t$, then only the pair $(a_{k\ell}, a_{kt})$ is included in $\mathcal{P}$.
- two cells $C_{k,\ell}$ and $C_{s,t}$ share a row, i.e., $\ell = t$, then we look at the sign (direction) associated with this row and the relative position of the two cells within the row.
  - If $r_\ell = 1$ (i.e., the row is oriented from bottom to top) and $k < s$ (the first of the two cells is to the left of the second one), then only the pair $(a_{s\ell}, a_{k\ell})$ is included in $\mathcal{P}$.
  - If $r_\ell = 1$ and $k > s$, then only the pair $(a_{k\ell}, a_{s\ell})$ is included in $\mathcal{P}$.
  - If $r_\ell = -1$ (i.e., the row is oriented from top to bottom) and $k < s$, then only the pair $(a_{k\ell}, a_{s\ell})$ is included in $\mathcal{P}$.
  - If $r_\ell = -1$ and $k > s$, then only the pair $(a_{s\ell}, a_{k\ell})$ is included in $\mathcal{P}$.

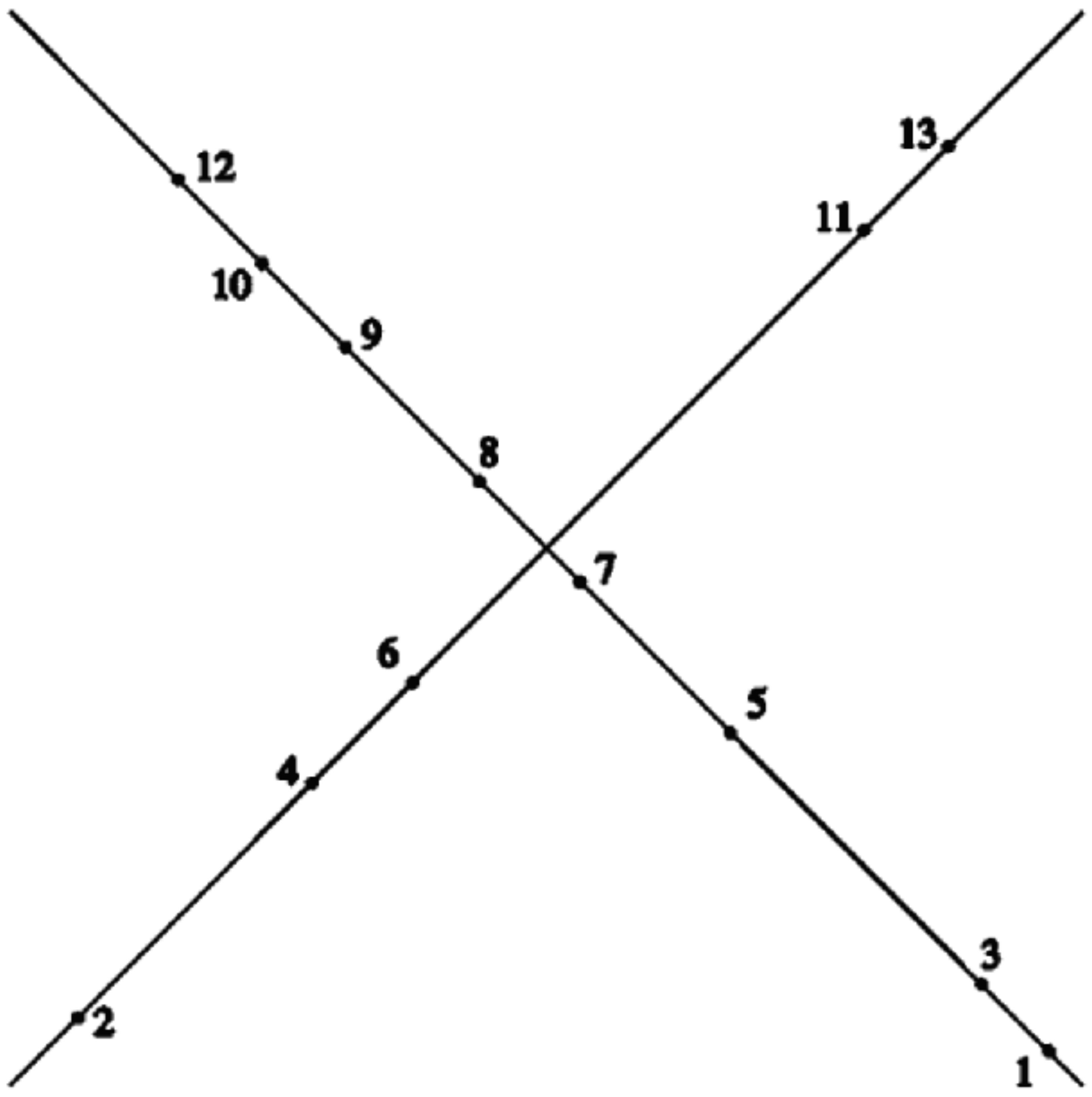
Alecu, Bogdan; Lozin, Vadim; de Werra, Dominique; Zamaraev, Viktor

Letter graphs and geometric grid classes of permutations: characterization and recognition.

Discrete Appl. Math. 283, 482-494 (2020).

Geometric grid classes

Defined by example



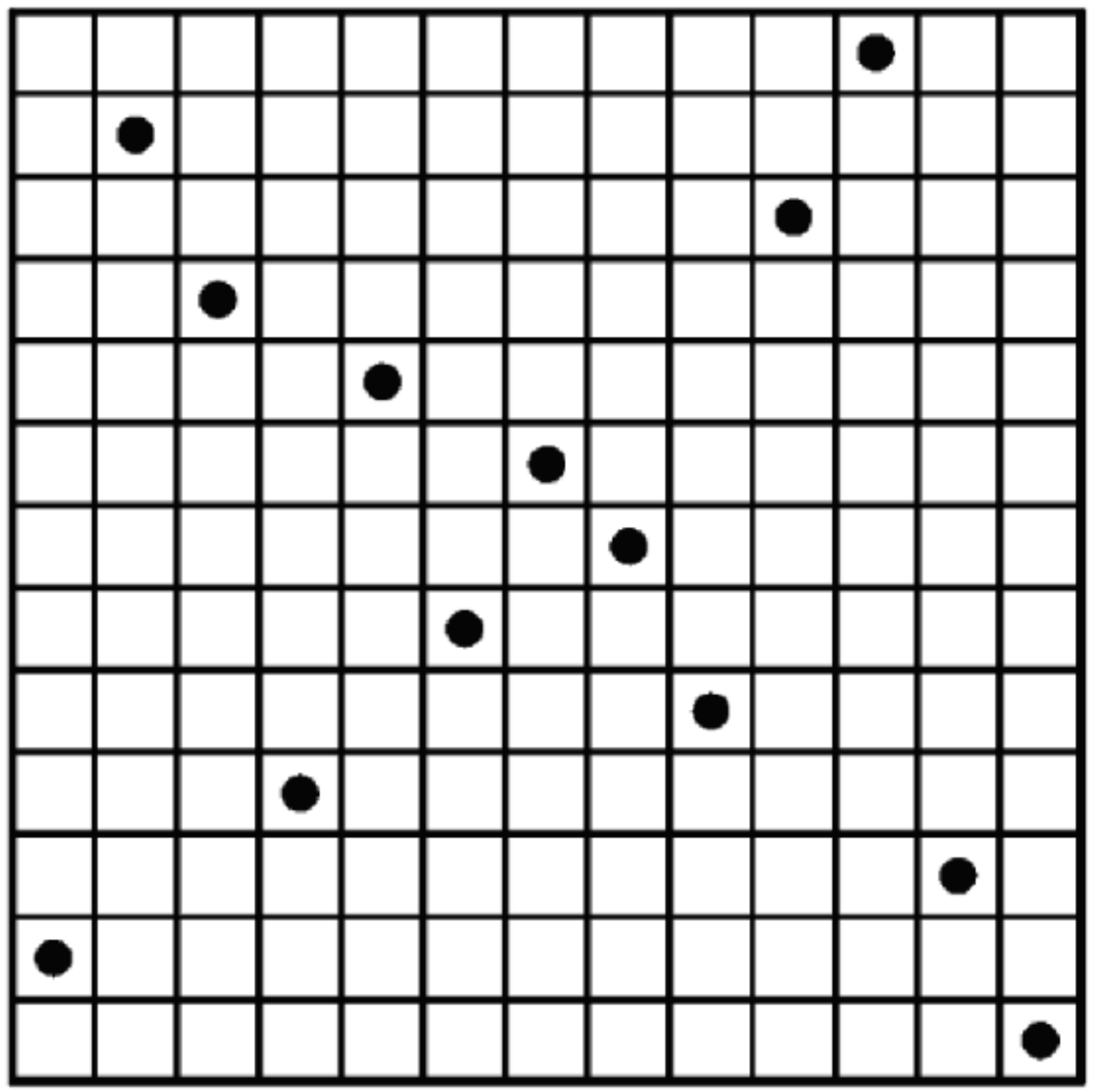


Figure 2: A permutation in the X-class, $\pi = (2, 12, 10, 4, 9, 6, 8, 7, 5, 11, 13, 3, 1)$.

THEOREM 1.1. *The permutation class C is geometrically griddable if and only if the corresponding graph class G_C has bounded lettericity.*

PROPOSITION 6.7. *Suppose that the permutation π has only trivial monotone intervals, that $\pi \in \mathrm{Grid}(M)$ for a $0/\pm 1$ matrix M of size $t \times u$, and that $G_\pi \cong \Gamma_D(w)$ where $|\Sigma| \leq r$, $D \subseteq \Sigma^2$, and $w \in \Sigma^*$. Then there is a $0/\pm 1$ matrix $M^{\times}$ of size at most $t(1+2ur) \times u(1+2tr)$ for which $\pi \in \mathrm{Geom}(M^{\times})$.*

Alecu, Bogdan; Ferguson, Robert; Kanté, Mamadou Moustapha; Lozin, Vadim V.; Vatter, Vincent; Zamaraev, Victor
Letter graphs and geometric grid classes of permutations.
SIAM J. Discrete Math. 36, No. 4, 2774-2797 (2022).

What is the largest lettericity of an n-vertex graph?

By a counting argument we now improve this bound to $l(n) > 0.707n$, provided that n is large enough.

Theorem 7. *For each $\alpha < (\sqrt{2}/2)$ there is an N such that for all $n > N$ there are n-vertex graphs G with $l(G) > \alpha n$.*

Therefore

$$2^{\binom{n}{2}} \leqslant n!k^n 2^{k^2} \leqslant n^n(\alpha n)^n 2^{(\alpha n)^2}.$$

Taking base 2 logarithms we have

$$(\tfrac{1}{2} - \alpha^2)n^2 \leqslant 2n \lg n + (\tfrac{1}{2} + \lg \alpha)n.$$

Since $1/2 > \alpha^2$ this is impossible when n is large. $\square$

Petkovšek, Marko
Letter graphs and well-quasi-order by induced subgraphs.
Discrete Math. 244, No. 1-3, 375-388 (2002).

What is the *expected* lettericity of an n-vertex graph?

We *can* just give each vertex its own letter (n-vertex graph has lettericity at most n).

We can "save" at least one letter by have the first and last vertex encoded by the same letter:

abcd... a

Save two letters?

What is the expected lettericity of an n-vertex graph?

Proposition 1. *Suppose G is a graph with n vertices containing an induced subgraph H with $2k$ vertices that is a k-letter graph for a word of the form*

$$w = \ell_1\, \ell_2 \ldots \ell_k\, \ell_{\pi(1)}\, \ell_{\pi(2)} \ldots \ell_{\pi(k)}$$

for some permutation π of $\{1, \ldots, k\}$. Then, w can be extended to a lettering of G by inserting new letters into the middle of w, and thus $\ell(G) \leqslant n - k$.

Mandrick and Vatter, Bounds on the lettericity of graphs

Electronic J. Comb. (2024), P4.53

What is the expected lettericity of an n-vertex graph?

Theorem 3. *For every k and each graph G on $n \geqslant 2(k-1) + 2^{2(k-1)} + 1$ vertices, G has an induced subgraph with $2k$ vertices that is a k-letter graph on the word*

$$w = \ell_1\, \ell_2 \ldots \ell_k\, \ell_k \ldots \ell_2\, \ell_1.$$

Thus, $\ell(G) \leqslant n - k$ by Proposition 1.

Corollary. *For every n-vertex graph G, we have*

$$\ell(G) \leq n - \frac{1}{2}\log_2 n$$

Mandrick and Vatter, Bounds on the lettericity of graphs

Electronic J. Comb. (2024), P4.53

What is the expected lettericity of an n-vertex graph?

Proposition. *For almost all graphs G, no three vertices can be encoded by the same letter in a lettering of G.*

$$w = w_1 \, a \, w_2 \, a \, w_3 \, a \, w_4.$$

Mandrick and Vatter, Bounds on the lettericity of graphs

Electronic J. Comb. (2024), P4.53

What is the expected lettericity of an n-vertex graph?

Theorem. For almost all graphs G on n vertices,

$$\ell(G) \geq n - (2\log_2 n + 2\log_2 \log_2 n)\,.$$

Theorem. For all graphs G on n vertices,

$$\ell(G) \leq n - \frac{1}{2}\log_2 n\,.$$

Mandrick and Vatter, Bounds on the lettericity of graphs

Electronic J. Comb. (2024), P4.53

What is the expected lettericity of an n-vertex graph?

Theorem. For almost all graphs G on n vertices,

$$\ell(G) \geq n - (2\log_2 n + 2\log_2\log_2 n)\,.$$

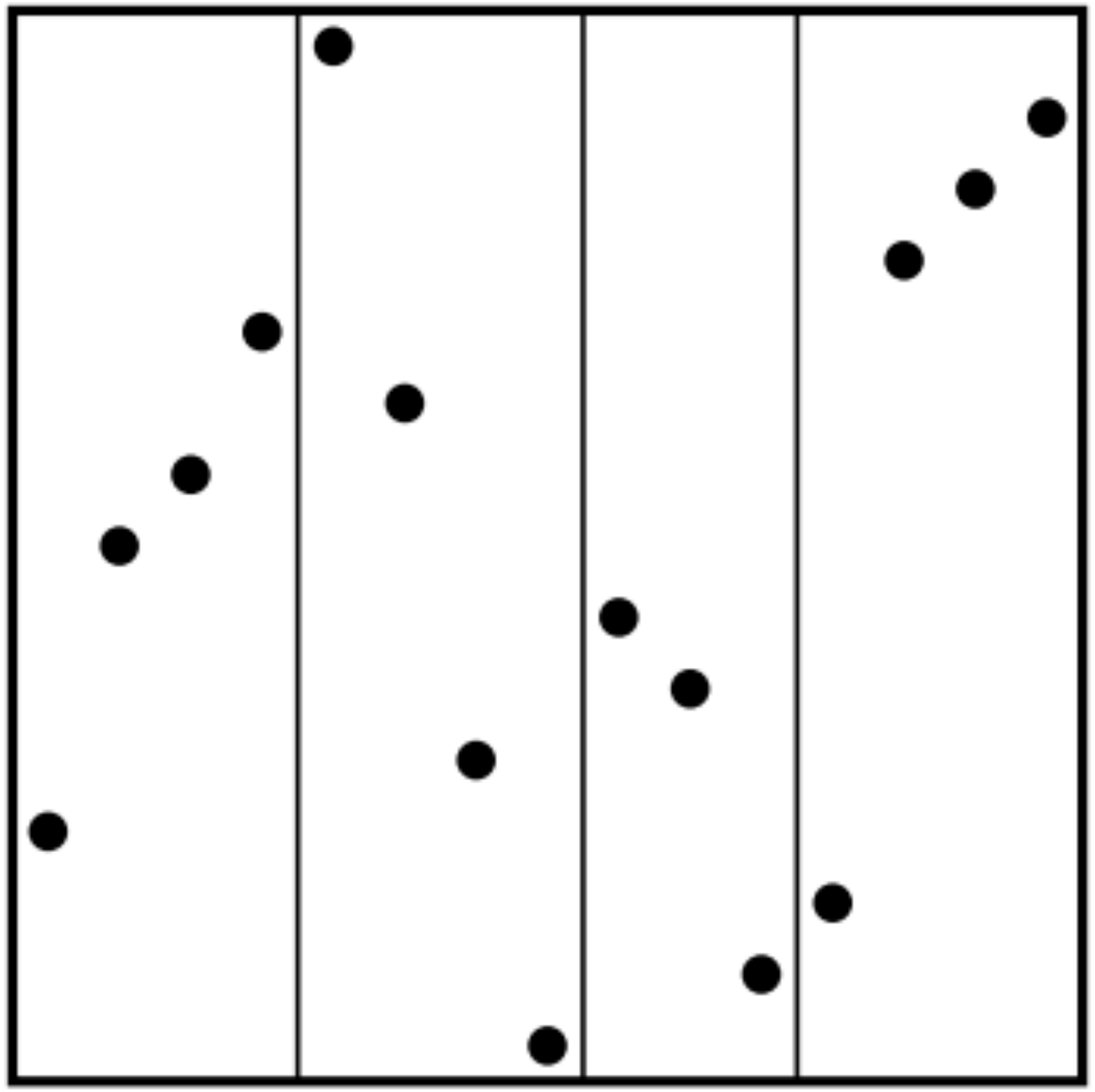
Mandrick and Vatter, Bounds on the lettericity of graphs

Electronic J. Comb. (2024), P4.53

What is the expected "lettericity" of permutation of length n?

Mandrick, in preparation: $\leq 0.4132n$,

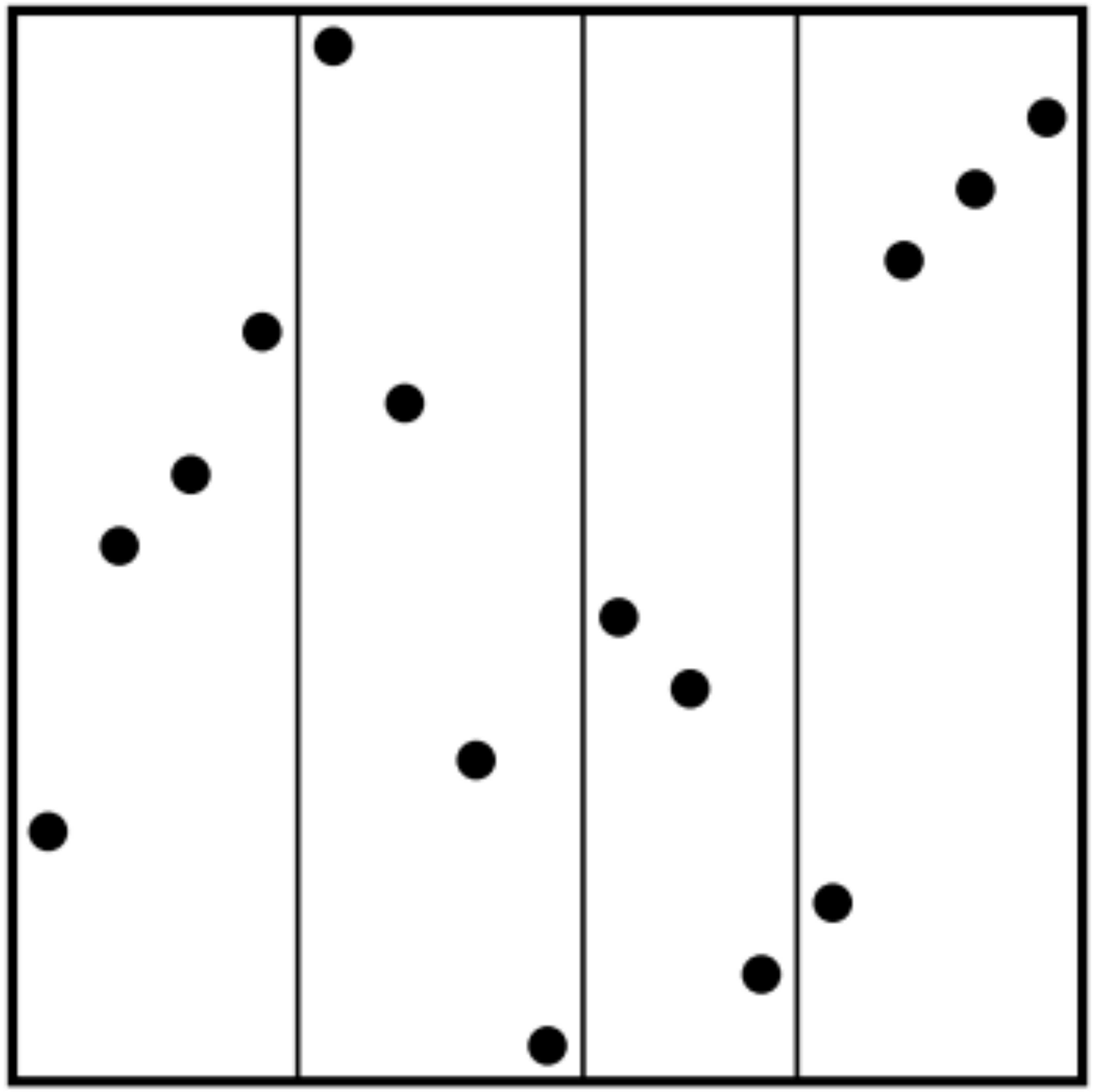
but that is just using "skinny grid classes".



What is the expected "lettericity" of permutation of length n?

Mandrick, in preparation: $\leq 0.4132n$,

but that is just using "skinny grid classes".



Thank you.