Enumerating 1324-avoiders with few inversions

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Permutation Patterns 2025

Based on joint work with

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1324

Wilf classes of length 4 patterns

Gessel (1990):

$$\mathrm{av}_n(1234) = 2\sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}^2 \frac{3k^2 + 2k + 1 - n - 2nk}{(k+1)^2(k+2)(n-k+1)}.$$

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Bóna (1997):

$$\begin{aligned}\mathrm{av}_n(1342) = {} & (-1)^{n-1}\cdot\frac{7n^2-3n-2}{2} \\ & + 3\sum_{k=2}^{n}(-1)^{n-k}\cdot 2^{k+1}\cdot\frac{(2k-4)!}{k!(k-2)!}\cdot\binom{n-i+2}{2}.\end{aligned}$$

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Big Problem. Say anything about $\mathrm{Av}(1324)$.

Known bounds for the Stanley–Wilf limit

$$L(1324) := \lim_{n\to\infty} \mathrm{av}_n(1324)^{1/n}:$$

	Lower	Upper
Bóna (2004)		288
Bóna (2005)	9	
Albert–Elder–Rechnitzer–Westcott–Zabrocki (2006)	9.35	
Claesson–Jelínek–Steingrímsson (2012)		16
Bóna (2014)		13.93
Bóna (2015)		13.74
Bevan (2015)	9.81	
Bevan–Brignall–Elvey Price–Pantone (2020)	10.27	13.5

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An innocent conjecture

Inversion = 21-pattern.

$$\mathrm{Av}_n^{k}(p) = \{\pi \in \mathrm{Av}_n(p) : \mathrm{inv}(\pi) = k\}\,, \qquad \mathrm{av}_n^k(p) = |\mathrm{Av}_n^k(p)|.$$

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Conjecture (Claesson–Jelínek–Steingrímsson, 2012)**.**

$$\mathrm{av}_n^k(1324) \leqslant \mathrm{av}_{n+1}^k(1324)$$

for all k and n, i.e. 1324 is *inversion monotone*.

Corollary. $L(1324) < 13.002$.

Numbers $\mathrm{av}_n^k(1324)$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1																			
2	1	1																		
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4	1	2	5	6	5	3	1													
5	1	2	5	10	16	20	20	15	9	4	1									
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1				
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521
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12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312
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14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786
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CJS conjecture: increasing

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Claesson–Jelínek–Steingrímsson (2012) finds the blue sequence.

Linusson–V. (2025) proves the conjecture in the red region.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	1																		
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6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	303	240	165	98	49	20
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	956	1237	1463	1583	1570	1421
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	724	1237	1961	2898	4012	5213
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	317	569	1028	1795	3034	4867
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	172	300	517	866	1435	2394
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	316	526	876	1397
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	550	904
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

Row differences

CJS conjecture: nonnegative

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14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
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15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

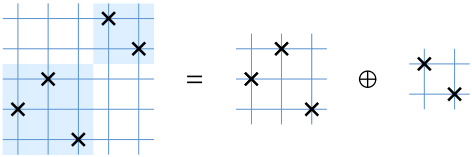
A constant sequence appears in the red region.

We can find its generating function.

What?

Decomposable permutations

Permutations can be decomposed with respect to the *direct sum*:



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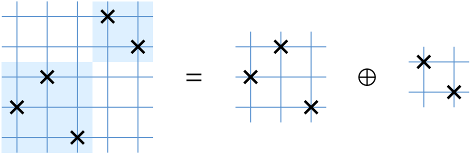
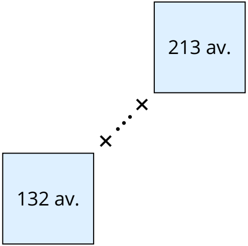


Fact. A decomposable $\pi \in S_n$ avoids 1324 if and only if

$$\pi = \sigma \oplus \mathrm{id} \oplus \tau,$$

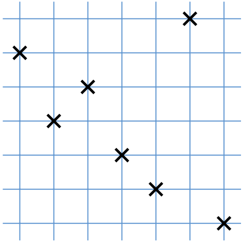
where $\sigma \in \mathrm{Av}(132)$, $\tau \in \mathrm{Av}(213)$.



Fact. If $n \geqslant k+1$, then

$$\mathrm{av}_n^k(132) = p(k),$$

the number of integer partitions of k.



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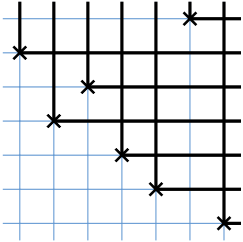

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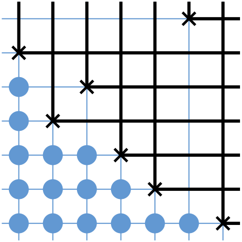

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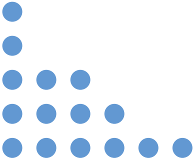

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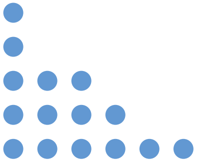

Fact. We have

$$\mathrm{inv}(\pi) + \mathrm{comp}(\pi) \geqslant |\pi|,$$

and thus π is decomposable if $|\pi| \geqslant \mathrm{inv}(\pi) + 2$.

The limit sequence

Combining the facts. If $n \geqslant k + 2$, then

$$\mathrm{av}_n^k(1324) = \sum_{i=0}^{k} p(i)p(k-i) = [x^k]\big(P(x)^2\big).$$

This is the blue sequence

1, 2, 5, 10, 20, 36, 65, 110, 185, 300, 481, 752, 1165, . . .

from the table!

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The estimate $p(k) \leqslant e^{\pi\sqrt{\frac{2}{3}k}} \approx 13.002^{\sqrt{k}}$ gives the new bound for $L(1324)$.

1324

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Question. 1324-avoiders with very few inversions are decomposable. What if we allow slightly more inversions?

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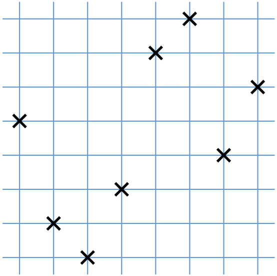
Definition (Linusson–V. 2025)**.**
$\pi \in S_n$ is *almost decomposable* if it is indecomposable, but

$$\operatorname{comp}(\pi \setminus e) \geqslant 2.$$

for at least one $e \in \{1, n, \pi_1, \pi_n\}$.



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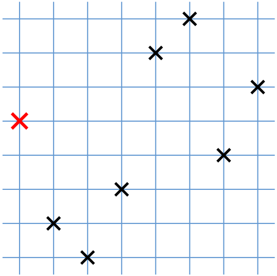

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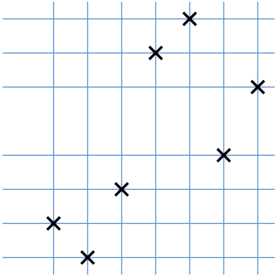

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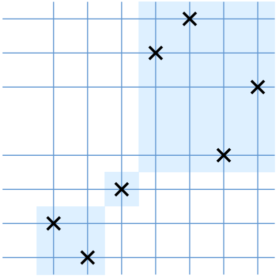

Theorem (Linusson–V. 2025). If $n \geqslant \frac{k+7}{2}$, then all permutations in $\mathrm{Av}_n^k(1324)$ are decomposable or almost decomposable.

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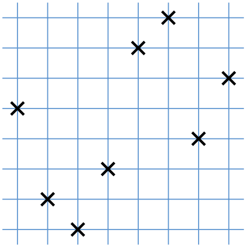
We can now build an injection $\mathrm{Av}_n^k(1324) \to \mathrm{Av}_{n+1}^k(1324)$:



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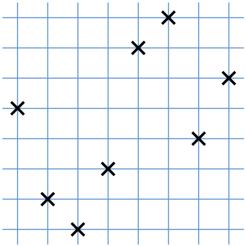

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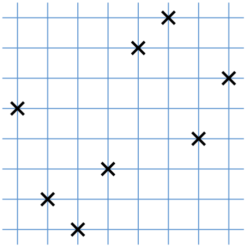

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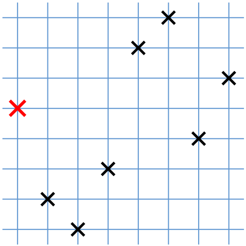

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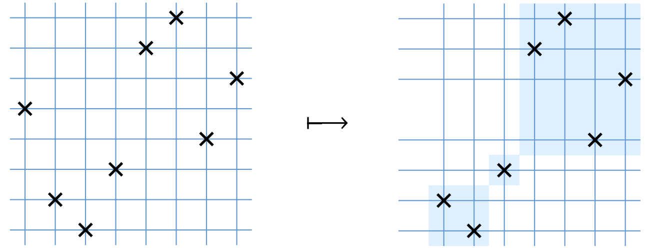

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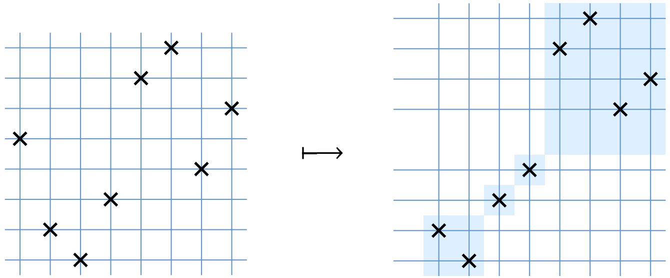

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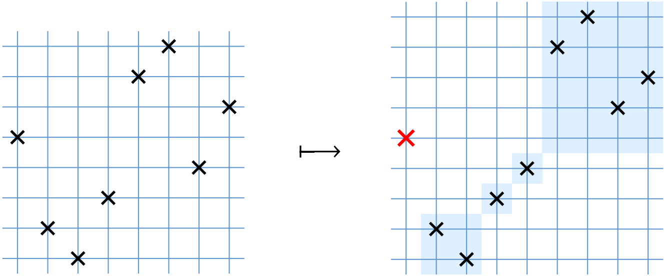

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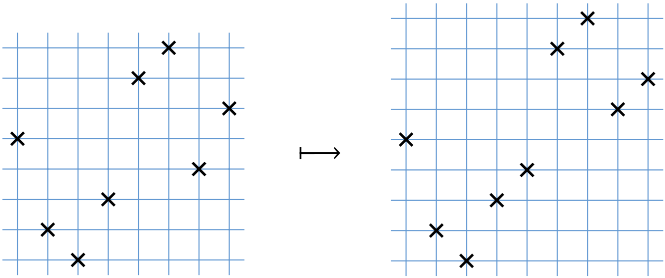
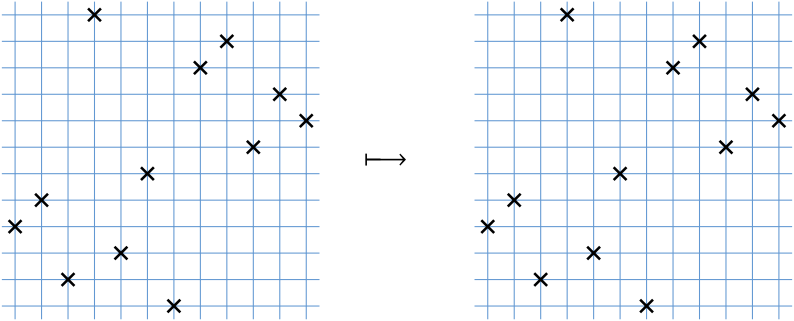


We extend the map symmetrically. Another example:



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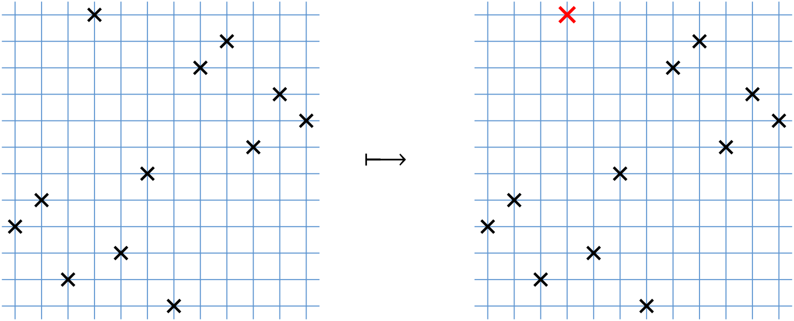

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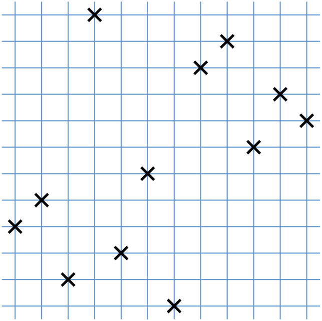

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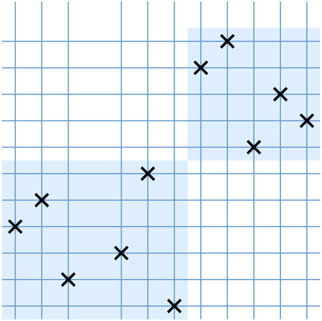

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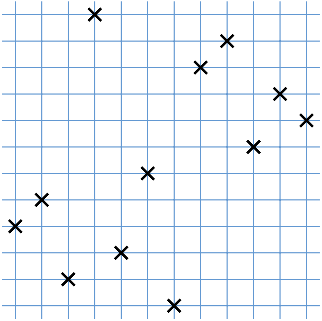

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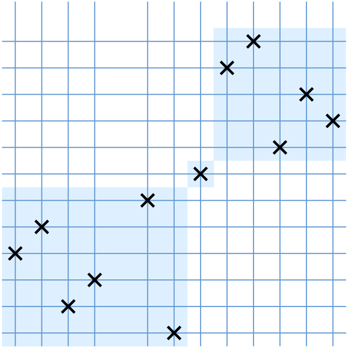

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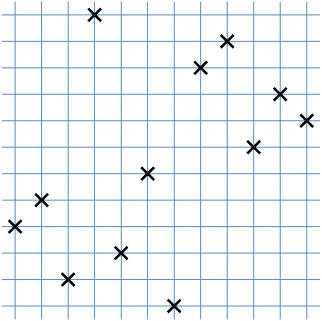

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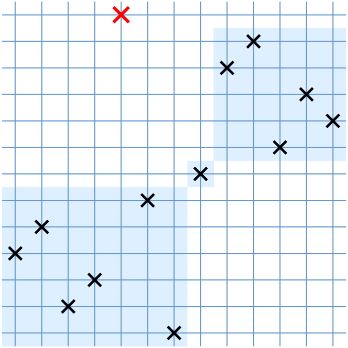

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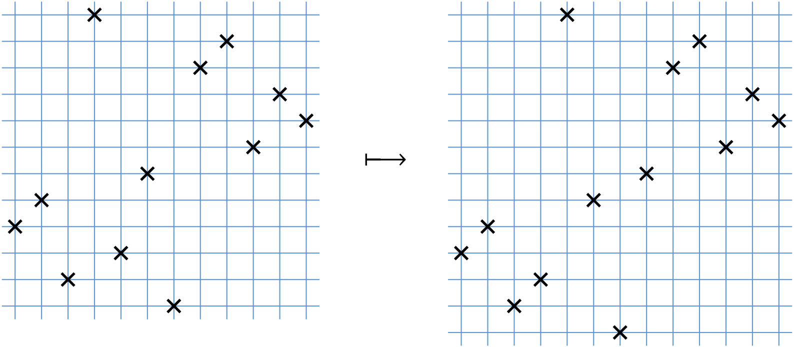

The whole thing turns out to be injective.

Theorem (Linusson–V. 2025)**.** For all $n \geqslant \frac{k+7}{2}$,

$$\mathrm{av}_n^k(1324) = a(k) - 4a(k-n+1) - 6\sum_{i=0}^{k-n} a(i),$$

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Corollary. For all $n \geqslant \frac{k+7}{2}$,

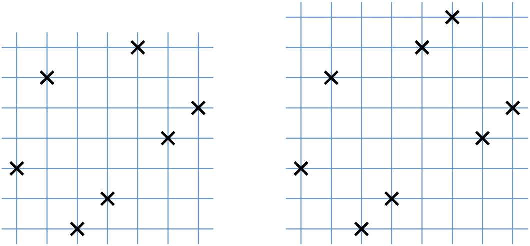
$$\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324) = 4a(k-n+1) + 2a(k-n) \geqslant 0,$$

i.e. the conjecture holds.

Can we do more?

No

Bad news. $\mathrm{Av}_n^k(1324)$ contains non-almost decomposable permutations for all $n < \frac{k+7}{2}$.



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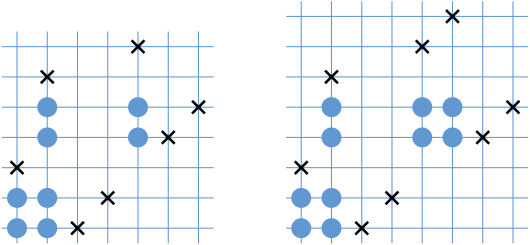

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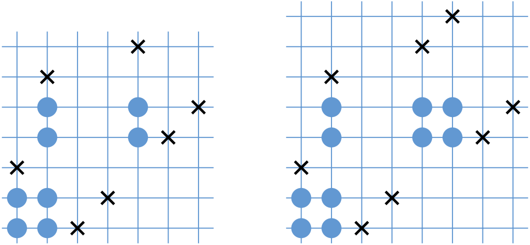

Bóna used the same counterexample for a different purpose.

But we can look at data

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1																					
2	1	1																				
3	1	2	2	1																		
4	1	2	5	6	5	3	1															
5	1	2	5	10	16	20	20	15	9	4	1											
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1						
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20	6	1
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441	1173	866
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654	7559	8225
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521	15012	19067
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915	18976	25563
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312	21171	28973
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216	22632	31266
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786	23568	32768
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118	24138	33728
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302	24470	34298
17	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5818	8456	12192	17402	24654	34630

$$P(x)^2, \qquad \text{where } P(x) = \sum_{k\geqslant 0} p(k)x^k$$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	1																				
2	0	1	2	1																		
3	0	0	3	5	5	3	1															
4	0	0	0	4	11	17	19	15	9	4	1											
5	0	0	0	0	4	12	31	52	70	76	67	49	29	14	5	1						
6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	303	240	165	98	49	20	6	1
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	956	1237	1463	1583	1570	1421	1167	865
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	724	1237	1961	2898	4012	5213	6386	7359
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	317	569	1028	1795	3034	4867	7453	10842
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	172	300	517	866	1435	2394	3964	6496
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	316	526	876	1397	2195	3410
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	550	904	1461	2293
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570	936	1502
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570	960
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184

$$(4+2x)P(x)^2$$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1	1																			
2	0	2	3	4	3	1																
3	0	0	1	6	12	16	14	9	4	1												
4	0	0	0	0	1	14	33	55	67	63	48	29	14	5	1							
5	0	0	0	0	0	-2	-2	17	58	124	205	273	304	289	235	164	98	49	20	6	1	
6	0	0	0	0	0	0	0	-6	-15	1	47	161	350	623	934	1223	1418	1472	1372	1147	859	573
7	0	0	0	0	0	0	0	0	1	-8	-33	-46	-36	52	281	724	1435	2429	3643	4965	6192	7106
8	0	0	0	0	0	0	0	0	0	0	4	-4	-35	-80	-155	-209	-166	136	855	2240	4456	7695
9	0	0	0	0	0	0	0	0	0	0	0	0	8	6	-17	-52	-162	-360	-640	-903	-957	-424
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	16	9	10	-38	-199	-554	-1175
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16	24	28	64	98	97
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20	32	41	96
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	24	40
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Not done yet …

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	2	3	2	1																		
2	1	4	9	12	11	8	4	1														
3	0	0	0	8	21	39	53	54	44	28	14	5	1									
4	0	0	0	-2	-3	3	25	69	138	210	256	260	221	159	97	49	20	6	1			
5	0	0	0	0	0	-4	-13	-16	-11	37	145	350	630	934	1183	1308	1274	1098	839	567	337	174
6	0	0	0	0	0	0	1	-2	-18	-47	-83	-109	-69	101	501	1206	2225	3493	4820	5959	6686	6845
7	0	0	0	0	0	0	0	0	3	4	-2	-34	-119	-261	-447	-588	-580	-189	813	2730	5806	10145
8	0	0	0	0	0	0	0	0	0	0	4	10	18	28	-7	-151	-474	-1039	-1812	-2664	-3296	-3225
9	0	0	0	0	0	0	0	0	0	0	0	0	4	10	26	62	124	161	86	-272	-1182	-2872
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	8	19	54	136	296	527	770
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	8	13	32	84	208
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	8	14	24
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	8
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Conjecture. $(4 - 6x^2 + 2x^4)P(x)^2$

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Thank you!

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